# STATISTICAL TABLES

*By Ronald A. Fisher*

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THE THEORY OF INBREEDING. 1949. Oliver and Boyd Ltd., Edinburgh.

*By Frank Yates*

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# STATISTICAL TABLES

## FOR BIOLOGICAL, AGRICULTURAL AND MEDICAL RESEARCH

BY

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# PREFACE TO FIRST EDITION

THE problems with which an active statistical department may have to deal, require, if their solutions are to be made widely accessible, a great variety of special tables. A number of those in the present book are familiar to statisticians, and are already widely used. In presenting them in a convenient form, the opportunity has been taken to supplement them with a selection of others, chosen as likely also to be of value, and not accessible elsewhere. The volume is completed with a number of tables of standard functions of general utility. The experience of the authors of the problems arising in practical research, is the basis of the selection, from among those tables which from time to time have been computed for special purposes.

To perform their compilation, the main work has been to complete the tables in a form suitable for the general user, and, what is by no means easy, to ensure their entire accuracy. In addition we have been principally concerned with the design of printing and arrangement, which shall be suitable both for constant and for occasional users. Comments from both classes of reader will therefore be especially valued.

We are indebted to all workers at the Galton Laboratory, and at the Statistical Laboratory, Rothamsted, and especially to W. L. Stevens, for assistance in the completion of our task.

R. A. F.
F. Y.

*August* 1938

# PREFACE TO FOURTH EDITION

THE second edition contained as new material Tables V1 and V2 for the test of significance of the differences between two means obtained by different methods; Table VIII1, which supplies solutions of the type of problem illustrated in Examples 1, 2 and 4, more direct than those previously available; and Table VIII2, which gives densities of organisms estimated by the dilution method. Additions to the third edition consisted of Table XI1 (due to Dr D. J. Finney) which gives the modified probit weighting coefficients necessary for dosage mortality tests involving deaths among controls; the inclusion of the 10 per cent. points of $z$ and $e^{2z}$ (based on the tables of the incomplete Beta function of Miss Catherine M. Thompson, for which we are indebted to Prof. E. S. Pearson and Dr V. G. Panse), in Table V; Table XIV1, which gives a method of scoring linked data from intercrosses of double heterozygotes; the extension of Table XXIII of orthogonal polynomials for values of $n'$ from 53 to 75 (due to Mr V. Satakopan); and the extension of Table XXVI of natural logarithms to cover the range ·100 to 1·00.

The combinatorial solutions of Table XVII were considerably extended and simplified in presentation in the second edition by the more extensive use of solutions of the cyclic type, and the corresponding section of the introduction was expanded to give an account of Youden's Squares, and of the method of utilising information, formerly discarded, from comparisons between blocks. Further new solutions were added in the third edition, and some further modifications have been made in the fourth edition.

Table VIII was improved in the third edition so as to allow for interpolation for values of the observed event above 10; Table XI, of weighting coefficients and probit values for adjustments of special accuracy, was recast in order to simplify its use; Table XXXIV of constants, weights and measures, etc., was slightly revised; and a small positive bias was removed from the 1 per cent. values of Table V1 for the significance of the difference between two means.

The following new material has been included in the fourth edition: Table VIII3, which provides a test for the existence of a periodic component; Table XIV2, which gives a set of functions termed "the segmental functions" which serve to specify the frequencies of non-recombinant and recombinant gametes in terms of the metrical positions of the centromere, a series of markers and the terminus of the chromosome arm; and Tables XXXIII1 and XXXIII2, which contain random permutations of 10 and 20 numbers respectively and are of use in the construction of experimental arrangements.

In addition a further table of the normal integral, Table VIII4, has been added. In this Table the deviation from the mean of the distribution is taken as the argument. The section of the introduction on dosage mortality tests involving a natural death rate has also been re-written. (We are indebted to Dr D. J. Finney for this.)

We have from time to time received a number of requests for the inclusion of simple tables, useful for special fields of work, or giving approximate tests or estimates, which can easily be derived from tables either given in this book or available elsewhere. Examples of such tables are: the factors required to obtain an estimate of the standard deviation from the mean range in samples from a normal distribution (derivable from Table XX—a note on the use of such a table is included in the introduction) ; the multiple infection transformation $y = 100(1 - e^{-x/100})$ (derivable from Table XXVI).

The inclusion of tables of this kind considerably increases the size of the book. Those giving approximate tests or estimates might also unduly encourage the use of such tests or estimates in fields in which they were not really appropriate. We have therefore considered it best to leave the compilation of such tables to the workers who require them. This has the advantage that they can then be arranged in the form which is most suitable for the particular purpose for which they are required.

For the convenience of owners of other editions, a list of errata has been included showing all the errors of any importance not previously discovered. Our thanks are due to those who have drawn our attention to these errors. Our especial thanks are also due to members of the Department of Statistics at Rothamsted, and particularly to Dr P. M. Grundy, for their pains in preparing the fourth edition for the press.

R. A. F.
F. Y.

16*th June* 1952

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# ERRATA TO THE THIRD EDITION

*The page numbers are those of the third edition; all these errors have been corrected in the fourth edition*

p. 21. *For* $\sqrt{(8\times 2{\cdot}5968\times 27/32)}$ *read* $\sqrt{(8\times 2{\cdot}5968\times 32/27)}$.

Table I, p. 31. $P = 0{\cdot}15$ : *for* 1·439521 *read* 1·439531. $P = 0{\cdot}56$ : *for* 0·582841 *read* 0·582842. $P = 0{\cdot}73$ : *for* 0·345125 *read* 0·345126.

Table III, p. 32, last column. *For* 12·941, 6·859, 5·405 *respectively, read* 12·924, 6·869, 5·408.

Table IV, p. 33, last column. *For* 16·268, 18·465, 20·517 *respectively, read* 16·266, 18·467, 20·515.

Table V, p. 34. $n_2 = 1$ : *for* 1·3097, 1·3400 *respectively, read* 1·3067, 1·3397. $n_1 = 24$ : *for* ·7452, ·1829 *respectively, read* ·7453, ·1831.

Table V, p. 35. $n_2 = 1$ : *for* 13·73, 14·59 *respectively, read* 13·64, 14·58.

Table V, p. 36. $n_1 = \infty, n_2 = 120$ : *for* ·0081 *read* ·0881.

Table V, p. 39. $n_1 = 6, n_2 = \infty$ : *for* 2·09 *read* 2·10.

Table V, p. 41. $n_1 = 1, n_2 = 2$ : *for* 98·49 *read* 98·50. $n_1 = 2, n_2 = 3$ : *for* 30·81 *read* 30·82. $n_1 = 8$ : *for* 5981, 99·36 *respectively, read* 5982, 99·37.

Table V, p. 42. *The following values were in error*: $n_1 = 1, n_2 = 3, 5, 7, 40, 120$; $n_1 = 2, n_2 = 5, 29, 60, 120$; $n_1 = 3, n_2 = 1, 40$; $n_1 = 4, n_2 = 6, 7, 8$; $n_1 = 5, n_2 = 6, 19$; $n_1 = 8, n_2 = 5, 29$; $n_1 = 12, n_2 = 21$; $n_1 = 24, n_2 = 5, 6, 8, 19, 21, 120$; $n_1 = \infty, n_2 = 3, 60, 120$. *For correct values see Table. The largest alteration to a value with* $n_2 \geqslant 10$ *amounts to* 0·0029.

Table V, p. 43. *The following values were in error*: $n_1 = 1, n_2 = 3, 5, 7$; $n_1 = 2, n_2 = 5, 120$; $n_1 = 4, n_2 = 6$; $n_1 = 5, n_2 = 19$; $n_1 = \infty, n_2 = 120$. *For correct values see Table. None of the alterations to values with* $n_2 \geqslant 10$ *exceeds* 2 *units in the last decimal place.*

Table XVII, p. 64, No. 11. *The instruction should read*, Use 19, taking any set of nine varieties occurring in the same block, and deleting that block. (*In the fourth edition a cyclic solution in two families is given.*)

Table XXII, p. 69. *The following values were in error*: $r = 12, s = 19, 21, 22, 23, 24, 25$; $r = 13, s = 23, 24, 25$; $r = 14, s = 24, 25$; $r = 15, s = 25$. *For correct values see Table. In no case do the first* 9 *digits require alteration.*

# INTRODUCTION

## TABLES I TO VII. TESTS OF SIGNIFICANCE BASED ON THE NORMAL DISTRIBUTION

Tables I to VII constitute a group of tables, based on the normal distribution, and now widely used in making tests of significance. The common uses of these tables are fully illustrated with numerical examples in Fisher's *Statistical Methods for Research Workers*, where they were first published. Further statistical problems soluble by the same tables are, however, constantly being discovered.

If $x$ is the deviation from the mean of a normal distribution having unit variance, the ordinate $z$, given in Table II, is obtained from the algebraic expression

$$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

The integral of this function gives the area of the curve, or the frequency of observations, between the limits of integration. The fraction of the frequency falling outside the limits $\pm x$ is, therefore,

$$P = 2\int_x^\infty z\,dx.$$

Table I gives the values of $x$ corresponding with various values of this fraction.

The distribution of $t$, due originally to "Student," is used to test the significance of a deviation when its standard error is *estimated from the data*; $t$ is the deviation divided by its estimated standard error. The algebraic expression for the ordinate of the distribution is

$$\frac{\frac{n-1}{2}!}{\frac{n-2}{2}!\,\sqrt{\pi n}} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{1}{2}(n+1)}}.$$

Table III gives the values of $t$ corresponding with various values of the probability of a random value falling outside the limits $\pm t$. The integer $n$ is the number of degrees of freedom available for the estimation of error.

The quantity $\chi^2$, used to test goodness of fit, homogeneity, and for a variety of other purposes, may be regarded as the sum of the squares of $n$ variables, which vary normally and independently about zero with unit variance. It is a useful single measure of the aggregate of a set of deviations from expectation. The element of frequency may be expressed as

$$df = \frac{1}{\frac{n-2}{2}!} (\tfrac{1}{2}\chi^2)^{\frac{1}{2}(n-2)} e^{-\frac{1}{2}\chi^2}\, d(\tfrac{1}{2}\chi^2).$$

When $n$ is even the probability of exceeding any given value $\chi^2$ is reducible to the partial sum of a Poisson series:

$$P = \int_{\chi^2}^\infty df = e^{-\frac{1}{2}\chi^2}\left\{1+(\tfrac{1}{2}\chi^2)+\frac{1}{2!}(\tfrac{1}{2}\chi^2)^2+\ \cdots\ +\frac{1}{\frac{n-2}{2}!}(\tfrac{1}{2}\chi^2)^{\frac{1}{2}(n-2)}\right\}.$$

The mean of the whole Poisson series is $\frac{1}{2}\chi^2$, the sum of the whole series is unity, and the number of terms in the partial sum is $\frac{1}{2}n$. In consequence the table of $\chi^2$ (Table IV), which gives the values of $\chi^2$ for various values of $P$, can be used to find the mean of a Poisson series of which the first $\frac{1}{2}n$ terms constitute a given fraction of the whole.

### *Example* 1

Four serious floods have occurred in a given region during a century's experience; supposing the numbers of floods experienced in different centuries are distributed in a Poisson series, find the mean number for which only one century in 20 would have four serious floods or less, and the mean number for which only one century in 20 would have 4 floods or more.

In the first case we need the first five terms of the Poisson series, so $n = 10$; the table shows the 5 per cent. value of $\chi^2$ is 18·307; the expectation would be 9·154 floods per century. If, on the contrary, only one century in 20 experienced 4 floods or more, the first four terms of the series would constitute 95 per cent. of the whole;

for $n=8$, the 95 per cent. $\chi^2$ is 2·733, corresponding with a frequency of 1·366 serious floods per century. Unless our experience has fallen in the upper, or the lower 5 per cent. range of the distribution, the expectation lies between 1·366 and 9·154 floods per century. More or less stringent probabilities might, of course, have been chosen. Table VIII1 has been constructed to make this test directly at the 10 per cent., $2\frac{1}{2}$ per cent. and $\frac{1}{2}$ per cent. limits. Entering the table with $a = 4$ and $N = \infty$, for instance, the $2\frac{1}{2}$ per cent. limits of the expectation are found to be 1·09 and 10·24.

---

In testing goodness of fit $\chi^2$ is simply calculated from the numbers observed and expected in each distinguishable class. If $m$ is the frequency expected and $a$ that observed, then

$$\chi^2 = S\,\frac{(a-m)^2}{m} = S\left(\frac{a^2}{m}\right) - S(a)$$

where the summation is taken over all classes. The number of degrees of freedom is the number of classes, the frequencies of which can be varied independently without violating any totals, sub-totals, etc., which may have been used in calculating the expectations. It is inadvisable to use classes with expectations less than 5, so that these will usually be grouped in larger classes in applying the test. (See, however, Table VIII.)

The quantity $z$ may be defined as the difference of one-half the natural logarithms of two different estimates of variance, one based on $n_1$ and the other on $n_2$ degrees of freedom. Table V gives values of $z$ for different values of the three variables $n_1$, $n_2$ and $P$. The corresponding values of the variance ratio, $e^{2z}$, are given on the opposite pages. We may usually consider a $z$ test as a test whether one estimate of a variance based on $n_1$ degrees of freedom significantly exceeds a second estimate based on $n_2$ degrees of freedom. We are indebted to C. G. Colcord and L. S. Deming (9) for the extension of the $z$ table to $P = \cdot 001$, to H. W. Norton and Iowa State College for $P = \cdot 2$, and to Catherine M. Thompson (40, 41) and the Department of Statistics, University College, London, and to V. G. Panse (42) for $P = \cdot 1$.

The distributions of $t$ and of $\chi^2$ are equivalent to special cases of the distribution of $z$. Thus when $n_1 = 1$, $t = e^z$ and $t^2$ for $n_2$ degrees of freedom is the variance ratio. Equally, when $n_2 = 1$, the variance ratio is the value of $1/t^2$ corresponding with the probability $1-P$.

When $n_2 = \infty$, the variance ratio is $\chi^2/n_1$ for $n_1$ degrees of freedom; and when $n_1 = \infty$ it is the value of $n_2/\chi^2$ for $n_2$ degrees of freedom, corresponding with the probability $1-P$.

From the value 6 onwards the values of $n_1$ tabulated are in harmonic progression; *i.e.* they are the numbers for which $24/n_1$ takes the values 4, 3, 2, 1 and 0. As the corresponding values of $z$ run smoothly other values of $n_1$ may be used by interpolating for the values of $24/n_1$. Similarly, the largest $n_2$ tabulated are those for which $120/n_2$ takes the values 4, 3, 2, 1 and 0. The $z$ values for any pair of numbers $n_1$ and $n_2$ can thus be found from the table. With lower accuracy the variance ratio may be similarly interpolated. For high values of $n_1$ and $n_2$ a useful approximation is given below the table for each value of $P$ tabulated.

In the table of $z$, like that of $\chi^2$, the probability given, and most frequently wanted, is that of a single tail of the distribution beyond the tabulated value. For the tables of $x$ and $t$ it is the sum of two equivalent tails.

In addition to its general use in the analysis of variance, the $z$ table is related to the partial sum of a binomial series in the same way as is that of $\chi^2$ to the Poisson series. The ratio of the sums of squares is $n_1/n_2$ times the variance ratio, or $n_1e^{2z}/n_2$. If $p : q$ is the ratio of the sums of squares, $P$ is the sum of the first $\frac{1}{2}n_1$ terms of the expansion of $(q+p)^{\frac{1}{2}(n_1+n_2-2)}$, and $1-P$ is the sum of the remaining $\frac{1}{2}n_2$ terms.

### *Example* 2

An event has occurred 3 times out of 14 trials. The probability of its occurring 3 times or less, is represented by the sum of the first four terms out of 15, hence $n_1 = 8$, $n_2 = 22$. If $P = 1$ per cent., the table gives a variance ratio 3·45; hence the ratio of the sums of squares is $3{\cdot}45 \times 8/22 = 1{\cdot}255 = p/q$. Thus $p = 1{\cdot}255/2{\cdot}255 = \cdot 557$. On the other hand, for the probability of its occurring 3 times or more, we need $n_1 = 24$, $n_2 = 6$, when the variance ratio is 7·31, and $q/p = 29{\cdot}24$, $p = 1/30{\cdot}24 = \cdot 0331$. Unless our experience has fallen in the upper or the lower 1 per cent. range of the distribution, the probability of occurrence lies between ·0331 and ·557.

This type of problem also may be examined at other levels of significance by the use of Table VIII1. Entering the table with $a = 3$, $p = 3/14 = 0{\cdot}214$, we find that the limits of expectation, at the probability level $P = \frac{1}{2}$ per cent., are 0·360 and 8·26, giving the corresponding limits of the probability of occurrence as ·0257 and ·590. If $P = 2\frac{1}{2}$ per cent. the limits are ·0466 and ·508.

If the event had not occurred at all in 14 trials, it is still possible to assign an upper limit to the probability. For $P = 2\frac{1}{2}$ per cent., for example, the upper limit to the probability is $1-\cdot 025^{\frac{1}{14}}$ or ·232, and for $P = \frac{1}{2}$ per cent. it is $1-\cdot 005^{\frac{1}{14}}$ or ·315, corresponding to upper limits of expectation of 3·25 and 4·41 respectively. The same result can be obtained by interpolation with respect to $1/N$ in Table VIII1.

---

Table VI gives the exact test of significance for inter-class correlation coefficients. This is equivalent to the $t$ test of the regression, which reduces to

$$t = \frac{r}{\sqrt{1-r^2}}\sqrt{n'-2}$$

for $n'-2$ degrees of freedom, where $n'$ is the number of pairs on which the correlation is based. The results often differ materially from the comparison of $r$ with its standard error estimated from the formula,

$$\sigma_r = \frac{1-r^2}{\sqrt{n'-1}}.$$

Other comparisons involving correlation coefficients are best carried out by the $z$ transformation provided by Table VII.

## TABLES V1 AND V2. SIGNIFICANCE OF DIFFERENCE BETWEEN TWO MEANS

Tables V1 and V2 give the test of significance, due originally to Behrens, for the difference of two means for which the errors are due to different causes, so that estimates of variance cannot properly be pooled. In general we need the percentile points of a distribution compounded of two Student distributions with $n_1$ and $n_2$ degrees of freedom. The ratio of the scale-factors $s_1$ and $s_2$ is given by

$$s_1/s_2 = \tan\theta,$$

and the difference is judged to be significant if it exceeds $d\sqrt{s_1^2+s_2^2}$.

For each level of significance the value of $d$ is required given the 3 known values $n_1$, $n_2$ and $\theta$. Table V1 by P. V. Sukhatme gives the 5 per cent. and 1 per cent. points for 15° intervals of $\theta$, and for the lattice of values 6, 8, 12, 24 and ∞ for $n_1$ and $n_2$. Four-point interpolation is sufficient for full accuracy.

Table V2 deals more fully with the marginal case $n_1 = \infty$, where the distribution is compounded of a normal component of known variance and a component of Student's type. This occurs when we compare estimates of a physical constant, one of which is based on a large quantity of data of low precision, while the other rests on few, but rather precise, observations. When the errors from these two estimates are of comparable magnitude, both can be taken into account by finding the angle, defined by

$$\tan\theta = \sigma/s,$$

while the significant value for the observed difference is

$$d\sqrt{s^2+\sigma^2}.$$

Thus for high angles the distribution tends to normality and for low angles it tends to Student's form.

Sukhatme's values for 1 per cent. have been revised to remove the very small positive bias, about ·001 in $d$, detected by comparison with the asymptotic calculation used in constructing Table V2.

*Example :—*

Let us suppose that a physical constant evaluated by a new method gives a mean of twelve determinations,

$$\bar{x} = 4{\cdot}77383,$$

and that the sum of the squares of the deviations of these values from their mean is

$$S(x-\bar{x})^2 = {\cdot}011580,$$

so that from 11 degrees of freedom the variance of the mean is estimated to be

$$s^2 = {\cdot}00008773,$$

and the estimated standard deviation of the mean is

$$s = {\cdot}009366.$$

For 11 degrees of freedom Student's $t$ is 2·718 at the 2 per cent. point, so that at this level of significance we should regard our observed mean as differing significantly from any value outside the limits

$$\begin{aligned} &\bar{x} \pm ts \\ &= 4{\cdot}77383 \pm {\cdot}02546 \\ &= 4{\cdot}74837,\ 4{\cdot}79929. \end{aligned}$$

If, now, numerous previous determinations, using a different method, have given the value,

$$\mu = 4{\cdot}744,$$

outside these limits, it would appear that there must be a real discrepancy between the results from the two methods used.

The comparison, however, is not complete, if we realise that $\mu$ also is not a perfectly precise determination, for it might be that though the value adopted is incompatible with the new observations, yet that both methods might be simultaneously compatible with some intermediate value of the constant.

Behrens' test enables this question to be answered, when the precision of $\mu$ is known, by averaging the probability of obtaining so high a deviation, when $\mu$ is assigned a frequency distribution instead of a fixed value. If, for example, the value of $\mu$ adopted, has a standard error, $\sigma$, based on a large number of degrees of freedom, with

$$\sigma = \cdot 00382,$$

we should find,

$$\tan \theta = \sigma / s = \cdot 4079,$$
$$\theta = 22^\circ \; 12' ;$$

and

$$d = \frac{\bar{x} - \mu}{\sqrt{s^2 + \sigma^2}} = \frac{\cdot 02983}{\cdot 010115} = 2 \cdot 949$$

Table V2 shows that at the 2 per cent. level of significance $d$ is only 2·704 at 20° and 2·637 at 30°, for 10 degrees of freedom. For 11 degrees of freedom lower values still would be significant. The discrepancy observed is therefore a significant one, even when the errors to which $\mu$ is liable are fully taken into account.

## TABLE VIII. TESTS OF SIGNIFICANCE FOR 2×2 CONTINGENCY TABLES

Contingency tables are normally tested for significant departures from independence by means of the $\chi^2$ test. Provided that more than one degree of freedom is involved this test is satisfactory, even when the expectation in each class is quite small; but with only a single degree of freedom (as in 2×2 contingency tables) the number of possible sets of values satisfying the given marginal totals is small, so that the discontinuity of the resultant distribution is marked, whereas the $\chi^2$ distribution, which is an approximation to it, is continuous. The effect of this discontinuity can be overcome by applying Yates's correction for continuity when calculating $\chi^2$. This is done by reducing by $\frac{1}{2}$ the values which are greater than expectation and increasing by $\frac{1}{2}$ those which are less than expectation.

The correction for continuity, which should be applied when the smallest expectation is less than 500, considerably increases the effective range of the ordinary $\chi^2$ test for a single degree of freedom; but when the smallest expectation is less than 100, the possible asymmetry of the actual distribution associated with given marginal totals becomes of importance. Such asymmetry implies that deviations of a given magnitude in one direction are less probable than those in the other.

Table VIII allows this asymmetry (and other slight deviations from the ordinary $\chi^2$ distribution) to be taken into account. It gives what are in effect the 2·5 per cent. and 0·5 per cent. points of $\chi_c$ (the square root of $\chi^2$ corrected for continuity) for each tail separately. Since there are three independent marginal totals in a 2×2 contingency table all possible contingency tables could only be completely covered by a three-dimensional table of $\chi_c$ for each level of significance. In general, however, the distribution of $\chi_c$ is not very different for sets of marginal totals which give the same smallest expectation, and for which the observed value in any one cell can assume the same number of values (*i.e.* tables in which the smallest marginal total is the same, this being one less than the possible number of values).

This property is utilised in Table VIII, where values of $\chi_c$ are shown for chosen values of the smallest expectation $m$, and of $p$, the smallest expectation divided by the smallest marginal total. (For a detailed account of the method of construction of this table see Yates (14).)

In the contingency table :—

$$\begin{array}{cc|c} x & y & a' \\ z & u & b' \\ \hline a & b & n \end{array}$$

where $a \leqslant b$, $a' \leqslant b'$ and $a \leqslant a'$, so that $a$ is the smallest marginal total, and $x$ is the number in the cell with smallest expectation, we have $m = aa'/n$, and $p = m/a = a'/n$. It will be noted that as $n$ increases with fixed $m$ and $p$, $a$ remains fixed and $a' : b' : n$ is constant. When $n$ is very large the distribution of $x$ tends to that of the binomial distribution $(p+q)^a$. When $n$ takes on its smallest possible value, which is that which gives the smallest integral value of $a' \geqslant a$, the "limiting contingency distribution" of $x$ is attained. The values of $\chi_c$ for these two extreme distributions are shown in Table VIII, the former in ordinary type and the latter in black type. In general the intermediate contingency distributions give intermediate values of $\chi_c$.

Furthermore, as $p$ tends to zero with fixed $m$, the expectations in all cells except that containing $x$ become large, and the Poisson distribution is approached, whatever the ratio of $a$ to $a'$. Thus for $p = 0$ there is only one set of values in Table VIII, representing the virtual 2·5 and 0·5 per cent. points of the Poisson distributions. Thus the table can be used to test the significance of departures from binomial distributions with known $p$ and Poisson distributions with known $m$, as well as for contingency tables.

It should be noted that when there is a zero entry the exact test of $2\times 2$ contingency tables involves a simple product of factorials easily calculated from Table XXX. When there is a small entry the exact calculation is never difficult and is often quicker than interpolation in any existing table. Such tables are intended only to save labour in more troublesome cases, and not to obviate the necessity of understanding the exact test.

It may also be noted that for a binomial distribution with $p = 0{\cdot}5$, $\chi^2$ corrected for continuity can be expressed in the form $\chi_c^2 = (a-b-1)^2/(a+b)$, where $a$ and $b$ are the numbers of successes and failures $(a>b)$. This gives a rapid preliminary test, based on the numbers of + and − signs, for quantitative data, such as the differences of paired observations, in which the probabilities of positive and negative deviations are judged to be equal. The values given in line 4 of each panel of Table VIII show that the continuity correction is particularly effective in this case.

### *Example* 3

The number of summers (April-September) with periods of over 20 days without rainfall at Rothamsted during this century and the last half of last century, are given in Table 1.

TABLE 1. SUMMERS WITH AND WITHOUT DROUGHT AT ROTHAMSTED

| | 1852-1900. | 1901-1937. | Total. |
|---|---|---|---|
| Experiencing drought . . | 4 | 9 | 13 |
| Without drought . . . | 45 | 28 | 73 |
| Total . . . | 49 | 37 | 86 |

Are the summers of the present century significantly more subject to droughts than those of the last?

$$\text{Here}\quad \chi^2 = \frac{(9\times 45 - 4\times 28)^2\times 86}{49\times 37\times 13\times 73} = 4{\cdot}291,$$

which would indicate significance, the 5 per cent. point for one degree of freedom (Table IV) being $3{\cdot}841$. Applying the correction for continuity, however, we obtain

$$\chi_c^2 = \frac{(8\tfrac{1}{2}\times 44\tfrac{1}{2} - 4\tfrac{1}{2}\times 28\tfrac{1}{2})^2\times 86}{49\times 37\times 13\times 73} = 3{\cdot}124.$$

Reference to Table IV indicates that significance is not attained. This may be confirmed by reference to Table VIII. We have

$$\chi_c = 1{\cdot}77,\quad \text{the smallest expectation } m = \frac{13\times 37}{86} = 5{\cdot}59,\quad p = \frac{m}{13} = \frac{37}{86} = 0{\cdot}43.$$

The observed set lies on the longer tail of the distribution since the number 9 with smallest expectation is greater than expectation. All the values of Table VIII ($P = {\cdot}025$) which bracket the required value lie between $1{\cdot}91$ and $2{\cdot}06$, so that significance is not attained. Notice, however, that had $p$ been nearly zero, with the same expectation in the smallest class but with observed set on the shorter tail, the observed value of $\chi_c$ would have indicated significance.

Linear interpolation in two dimensions in Table VIII gives the value of $1{\cdot}96$ for both binomial and limiting contingency distributions, indicating that at this particular point the ordinary $\chi^2$ table (or a table of the normal probability integral, such as is given in Table VIII4) gives a good estimate of the actual probability after the correction for continuity has been applied. The probability corresponding to $1{\cdot}77$ is ${\cdot}0384$ compared with the true value of ${\cdot}0389$ calculated by the procedure given in *Statistical Methods*, Section 21.02. A method of assessing the actual probability when the values of $\chi_c$ differ from those of $\chi$ for the same probability levels is illustrated in Example 5.

### *Example* 4

In the course of 500 spins of a roulette wheel, zero appears 24 times. What evidence is there that the wheel is biased in favour of the bank?

If the wheel is unbiased, the chance of zero occurring in any one spin is $\frac{1}{37}$, and zero may be expected 13·514 times on the average in 500 trials. The discrepancy is therefore 10·486, $\chi^2 = \frac{10{\cdot}486^2}{13{\cdot}514} + \frac{10{\cdot}486^2}{486{\cdot}486} = 8{\cdot}362$, and $\chi = 2{\cdot}892$. (Note that the class of non-occurrences as well as the class of occurrences must be included when calculating $\chi^2$.) Interpolation in Table VIII4 for the value 2·892 of the argument indicates that the probability of obtaining 24 or more zeros was only ·00191. Correcting for continuity, the discrepancy becomes 9·986, and $\chi_c^2 = 7{\cdot}584$, $\chi_c = 2{\cdot}754$, giving a probability of ·00294. Referring to Table VIII, however, for $m = 13{\cdot}5$ and $p = {\cdot}027$ (values in ordinary type) we see that the 0·5 per cent. point of $\chi_c$ is about 2·80, compared with the value of 2·576 given by normal theory. Thus the true probability is somewhat greater than ·005. (True value obtained by summing the binomial distribution = ·00559.) The omission of the correction for continuity introduces large errors, and even the use of the ordinary $\chi^2$ table gives a probability of one half the true value.

The problem may also be solved directly by the use of Table VIII1. Since $a$ is greater than 10 we must use the corrections given for $a = 11$ and over. $p = 24/500 = {\cdot}048$ and therefore the conventional standard error $\sqrt{\{a(1-p)\}} = 4{\cdot}79$. Multiplying this by the normal deviate for $P = \frac{1}{2}$ per cent. (single tail) gives $4{\cdot}79 \times 2{\cdot}576 = 12{\cdot}34$. The limits of expectation calculated from the conventional standard error are therefore 11·66 and 36·34. The corrections given by Table VIII1 for these limits are +1·70 and +2·80, giving the adjusted limits of 13·36 and 39·14, corresponding to probability limits ·0267 and ·0783. Since $\frac{1}{37} = {\cdot}0270$ the probability of obtaining as many as 24 zeros in 500 spins is just over ·005, agreeing with our previous solution.

### *Example* 5

Locate the upper and lower 2·5 and 0·5 per cent. points of a Poisson distribution with an expectation of 25.

If $x$ is the observed value and this is greater than expectation (*i.e.* lies on the longer tail) we have $\chi_c = (x-25-\frac{1}{2})/\sqrt{25}$. For $m = 25, p = 0$, $\chi_c = 2{\cdot}05$ for the 2·5 per cent. point, so that $x = 35{\cdot}8$. Hence a value of 36 or over has a smaller probability than ·025. For the 0·5 per cent. point, $\chi_c = 2{\cdot}76$, so that $x = 39{\cdot}3$, giving a value of 40 or over. Similarly, the lower 2·5 and 0·5 per cent. values of $x$ will be found to be 15·2 and 12·6, giving integral values of 15 and 12 respectively. Normal theory would give $x$'s about $\frac{1}{2}$ a unit in error for the 2·5 per cent. points and about a unit in error for the 0·5 per cent. points.

If the actual value of the probability of obtaining a value of say 14 or less is required, we may proceed as follows. At the lower 2·5 per cent. point $\chi_c$ is 1·86 and $\chi$ (normal theory) is 1·96, and at the lower 0·5 per cent. point $\chi_c$ is 2·37 and $\chi$ is 2·58. A value of 14 gives a $\chi_c$ of 2·10. Hence by linear interpolation, using $\chi_c$ as argument, the corresponding value of $\chi$ is

$$\chi = 1{\cdot}96 + \frac{2{\cdot}10-1{\cdot}86}{2{\cdot}37-1{\cdot}86}(2{\cdot}58-1{\cdot}96) = 2{\cdot}25.$$

Reference to Table VIII4 gives the corresponding probability as ·0122. (True value = ·0124.)

## TABLE VIII1. LIMITS FOR THE PROBABILITY OF AN EVENT

If an event is observed to occur $a$ times out of $N$, a lower limit $\pi_1$ can be assigned to the probability of this event such that if the probability were actually $\pi_1$ then an observed number of occurrences as great or greater than $a$ out of $N$ trials would only occur by chance with a frequency of $P$. Similarly an upper limit $\pi_2$ can be assigned such that if the probability were actually $\pi_2$ an observed number of occurrences as small or smaller than $a$ would occur with a frequency $P$. Corresponding to these probabilities $\pi_1$ and $\pi_2$ there are limits of expectation of the number of occurrences in $N$ trials, namely $\pi_1 N$ and $\pi_2 N$. These limits of expectation are tabulated in Table VIII1 for $P = 0{\cdot}1$, $0{\cdot}025$ and $0{\cdot}005$ against values of $a$ from 0 to 10 (with extension to higher values) and of $N$ from $2a$ to $\infty$. The values for $N = \infty$ gives the limits of expectation of the Poisson distribution. When $a > \frac{1}{2}N$, the table may be entered with $a' = N-a$. Examples of the use of the table have already been given (Examples 1, 2 and 4).

## TABLE VIII2. DENSITIES OF ORGANISMS ESTIMATED BY THE DILUTION METHOD

One of the methods available for estimating the number of organisms of a given type present in a medium is the dilution method. This method can be employed in cases in which the presence or absence of the organism can be determined, but in which counts cannot be made.

In the dilution method a series of suspensions of the organism, $s$ levels in all, is prepared, each of which is $a$ times as dilute as the preceding one. Each suspension is used to inoculate $n$ tubes with a known volume of the suspension, and the presence or absence of the organism is determined by suitable test after a period of incubation. The most accurate estimate of the number of organisms per tube at any given level is obtained by solving the equation of maximum likelihood, and a method of doing this has been discussed by Mather (62); but it has been shown (Fisher, 30) that 87·7 per cent. of the information is contained in the total number $X$ of fertile or $Y$ of sterile plates, counted without regard to level. If $\lambda$ is the number of organisms per tube at the highest concentration, the value of $\lambda$ for which the expected average number of sterile plates is equal to the observed number is given by the equation

$$Y = n(e^{-\lambda}+e^{-\lambda/a}+e^{-\lambda/a^2}+ \dots +e^{-\lambda/a^{s-1}})$$

Table VIII2 enables the solution of this equation to be obtained expeditiously for two-fold, four-fold or ten-fold dilution series of any length. All that is necessary is to calculate what may be called the mean fertile level, $x = X/n$, and the mean sterile level, $y = s-x$, and find the tabular value $K$ corresponding to the observed $x$ or $y$ for the appropriate number, $s$, of levels. The estimate of $\lambda$ is then given by

$$\log \lambda = x \log a - K.$$

In an extended series of dilutions the value of $K$ becomes constant for all values of $x$ and $y$ sufficiently distant from the ends of the distribution, except for a periodic component, which, however, is only of importance for the ten-fold series. This periodic component necessitates the special form of tabulation adopted for this series.

The average value of the variance of the mean fertile level is $\frac{1}{n}\frac{\log 2}{\log a}$ except at the ends of the series, where the value increases. The average value of the variance of $\log \lambda$ is $\frac{1}{n} \log 2 \,.\, \log a$, and this may be used in the two- and four-fold series in the range where the tabular entries are substantially constant. The variance of the mean fertile level is, however, more stable than the variance of $\log \lambda$. Closer approximation to the fiducial limits can therefore be found by using the average value $\frac{1}{n}\frac{\log 2}{\log a}$ of this variance, increasing and decreasing the observed mean fertile level by the appropriate multiple of the corresponding standard error. The table can then be re-entered to determine limiting values of $\log \lambda$, and thence, if required, of $\lambda$. This approximation is satisfactory except at the extreme ends of the range. For the same reason the mean fertile level should be used in preference to $\log \lambda$ when performing tests of significance, etc., on comparisons between different media. Standard errors should not be attached to $\lambda$.

The inclusion of tables for the four-fold and ten-fold series should not be taken to imply that the use of these series is ordinarily advisable. If an adequate number of tubes is available the two-fold dilution series should be used, with correspondingly fewer plates at each level, in preference to a four-fold or ten-fold series covering the same range. As is shown by the formula given above, the average variance of $\log \lambda$ in the centre of the range is the same for a given total number of plates, whatever the dilution factor, but the periodicity becomes appreciable for the more coarsely spaced series.

The number of dilution levels required depends primarily on the range of populations it is desired to cover. For any given population density the majority of the information is given by the levels at which the average number of organisms per tube is near 1. Thus in a two-fold dilution series in which one dilution level has an average of exactly one organism per tube the contributions of the various levels of an infinite series to the total amount of information are as follows:

| Average number of organisms per tube | 8 and more | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ and less |
|---|---|---|---|---|---|---|---|---|
| Percentage of information | 0·9 | 12·6 | 26·4 | 24·5 | 16·2 | 9·3 | 4·9 | 5·2 |

For any given population density, therefore, most of the information is contributed by five levels of the series. A series of 12 dilutions will therefore amply cover a range of 100 fold, and a series of 15 dilutions a range of 1000 fold. If more than one organism which can be detected in the same set of tubes is under investigation, then longer series may be required, the exact length depending on the relative numbers of the different types of organism.

When a large number of tests involving series of the same length and with the same number of tubes at each level have to be made, the experimenter may find it convenient to prepare a special table giving $\log \lambda$ and $\lambda$ directly for each possible value of the total number of fertile tubes. Such a table may be easily prepared from Table VIII2.

In practical work greater precision is often required in one part of the range than in the others. Thus the tests may have as their primary object the determination of whether the population densities are less than a given standard density, but at the same time a rough estimate of the actual population densities of samples differing widely from the standard may be of interest. In such a case the number of tubes should be increased at the levels for which the population per tube would be in the neighbourhood of 1 at the standard density, *e.g.* we might increase the number of tubes at levels having 4, 2, 1, ½ and ¼ organisms per tube at the standard density.

For the estimation of the population densities of dilution series of this type a special table will have to be prepared. If there are $n_1$ tubes at each of $s$ levels and an additional $n_2$ tubes at each of the levels from $r+1$ to $r'$, the average total number of sterile tubes is given by

$$Y = n_1y_1+n_2y_2 = n_1(e^{-\lambda}+e^{-\lambda/2}+ \; . \; . \; . \; +e^{-\lambda/2^{s-1}})+n_2(e^{-\lambda/2^r}+ \; . \; . \; . \; +e^{-\lambda/2^{r'-1}})$$

The table can be prepared directly by calculating the values of $Y$ for a series of values of log $\lambda$, thence determining by inverse interpolation the values of log $\lambda$ corresponding to integral values of $Y$. Alternatively Table VIII2 may be used to determine the values of $y_1$ and $y_2$ separately for a series of values of log $\lambda$, thence performing the inverse interpolation as before. Values of $y_2$ corresponding to values of log $\lambda$ falling outside the range of the table will in any case have to be computed directly, but this can be done rapidly, using the expansion of $y_2$ in powers of $\lambda$ for small values of $\lambda$.

The standard errors of estimates derived from a modified dilution series of this type naturally vary in different parts of the range. At the standard density, however, the standard errors will be those corresponding to an infinite series of $n_1+n_2$ tubes at each level, except for a slight increase due to the shortness of the subsidiary series.

### *Example* 5·1

Tests with potato flour containing rope spores (B. mesentericus) gave the following observations, using 5 tubes, each of 1 c.c., of dilutions containing 4, 2, 1, . . ., 1/128 g. per 100 c.c. (E. C. Barton-Wright's data).

| g. per 100 c.c. | Number fertile. | g. per 100 c.c. | Number fertile. |
|---|---|---|---|
| 4 | 5 | 1/8 | 3 |
| 2 | 5 | 1/16 | 2 |
| 1 | 5 | 1/32 | 2 |
| 1/2 | 5 | 1/64 | 0 |
| 1/4 | 4 | 1/128 | 0 |

The total number of fertile plates is 31, and the mean fertile level ($x$) is therefore 6·2. Hence $y$ is 3·8. The table gives $K = {}$·383. Hence

$$\log \lambda = 6{\cdot}2 \log 2 - {\cdot}383 = 1{\cdot}483 \; .$$
$$\lambda = 30{\cdot}4.$$

The number of organisms per gram is thus estimated to be 760.

The expected variance of the mean fertile level is 0·2. The fiducial limits of the number of tubes at levels ·025 to ·975 for $P$ are therefore $6{\cdot}2 \pm 1{\cdot}960 \times \sqrt{0{\cdot}2} = 5{\cdot}323$ and 7·077. Re-entering the table with these values, we obtain limits of 1·211 and 1·761 for log $\lambda$, or 407 and 1440 for the actual number of organisms per gram. The average standard error of log $\lambda$ is $\sqrt{{\cdot}0181} = {\cdot}135$, giving limits of $1{\cdot}483 \pm {\cdot}264 = 1{\cdot}219$ and 1·747 for log $\lambda$, or 415 and 1400 for the actual number, which may be compared with the more exact limits given by the first method.

## TABLE VIII3. SIGNIFICANCE OF LEADING PERIODIC COMPONENTS

If $u_r$ ($r = 1, 2, . . ., 2n+1$) constitute a random sample from a normally distributed population, the linear functions $S(a_ru_r)$ and $S(b_ru_r)$ define a Fourier component, if

$$a_r = \sqrt{\frac{2}{2n+1}} \cos \frac{2\pi pr}{2n+1}, \qquad b_r = \sqrt{\frac{2}{2n+1}} \sin \frac{2\pi pr}{2n+1}.$$

Values of $p$ from 1 to $n$ give $n$ pairs of components referable to different periods. If $A = S(a_ru_r)$ and $B = S(b_ru_r)$, then the sum of squares corresponding with the two degrees of freedom for any period is

$$X = A^2+B^2.$$

A general test for the existence of a periodic component is therefore provided by seeing whether the largest of the values of $X$ for the $n$ periods is greater than might be expected from chance causes. Table VIII3 shows at the 5 per cent. and 1 per cent. levels the fraction $g$ that the largest of the $n$ values of $X$ may bear to the total sum of squares, $S(u_r-\bar{u})^2$.

When the number of observations in the series is even, $2n+2$, the coefficients in $A$ and $B$ are

$$a_r = \frac{1}{\sqrt{n+1}} \cos \frac{\pi p r}{n+1}, \quad b_r = \frac{1}{\sqrt{n+1}} \sin \frac{\pi p r}{n+1}.$$

Values of $p$ from 1 to $n$ give $n$ values of $X$ corresponding with $n$ different periods. The table again gives the significance levels of the ratio $g$ that the largest $X$ bears to the sum of all $n$ values of $X$. In this case the sum of the $n$ values of $X$, which is the sum of squares for the $n$ periods, is equal to

$$S(u_r - \bar{u})^2 - \frac{1}{2n+2}(u_1 - u_2 + u_3 - \ . \ . \ . \ - u_{2n+2})^2.$$

The application of the table is, therefore, exceedingly simple although the subdivision of the sum of squares into its periodic components is somewhat tedious. The theoretical basis of the table was discussed by R. A. Fisher (55).

## TABLE VIII4. THE NORMAL PROBABILITY INTEGRAL

This table gives values of the integral of one tail of the normal distribution with unit standard deviation corresponding to given values of the deviation from the mean of the distribution. It is therefore virtually the inverse of Table IX, and is of particular use in those cases where values of the integral corresponding to exact hundredths of the standard deviation are required.

## TABLES IX TO XIV. THE PROBIT AND ANGULAR TRANSFORMATIONS

Many important classes of data may be interpreted on the supposition that a normal deviate is linearly dependent on some observable concomitant measurement, and that an observable frequency is that with which this deviate is exceeded in a normal distribution. For example, the frequency with which a weight is judged to be heavier than a standard increases with the actual weight used; the frequency with which a high jumper clears a rod decreases with the height at which it is placed; the frequency with which insects survive fumigation decreases with the concentration of the fumigant. In many such cases the relation between the frequency and measurement concerned is excellently represented by considering the normal deviate corresponding to the frequency to be a linear function of the measurement, or, more frequently in toxicology, of its logarithm, the measurement being the concentration of the toxin.

Table IX gives the deviate corresponding with each thousandth of the total frequency, which is exceeded by any given frequency; by means of the difference column it is simply used to ten-thousandths. To facilitate standard methods of calculation by making all values positive, 5 has been added to the deviate. The resulting values are known as " probits." It is obvious that the probits corresponding to complementary percentages, such as 40 per cent. and 60 per cent., must add to make 10. Beyond 95 per cent. the actual differences are inserted, and beyond 98 per cent. the tabulation is by ten-thousandths. By subtracting from 10, these advantages may be made available also at the beginning of the table.

Table X, simple quantiles of the normal distribution, gives the deviates corresponding with proper fractions. The proper fractions, up to $\frac{1}{2}$, having denominators up to 30, are here arranged in order of magnitude. When small numbers are involved this table enables the probits to be found more expeditiously than does Table IX.

### *Example* 6

What deviate is exceeded by 4/17 of a normal curve? The deviate given is 0·7215, so that if 4 animals survive out of 17 the probit is 5·7215. The same value may be obtained by expressing 13/17 as a decimal, ·7647, and referring to Table IX.

### *Dosage Mortality Tests*

When tests have been made on several concentrations it is usual to fit a straight line expressing the probit in terms of the logarithmic concentration. The weight assigned to each point will be the number of animals used in determining it multiplied by the weighting coefficient given in Table XI. It is best to draw a provisional line graphically, and obtain the weighting coefficients from the probits read from the graph. Some workers prefer to draw a provisional line by calculation, using weighting coefficients based on the probits empirically observed. This leads to some ambiguity when at any concentration all die (or all survive), for such observations nominally give an infinite probit with zero weight.

However the provisional line be obtained, for refined purposes, such as tests of significance, it should be replaced by one based on working probits, obtainable from Table XI. When all animals survive, the minimum working probit in the 2nd column corresponding to the expected probit in the 1st is used; when all die, the maximum working probit in the 4th column is used. Other cases are assigned values greater than the minimum working probit by the product of the proportion killed and the range (given in the 3rd column); the same quantity is obtained by decreasing the maximum working probit by the product of the proportion surviving and the range. The working probit is given a weight found by multiplying the total number of animals on which it is based by the weighting coefficient in the 5th column of the table. Any tolerably good provisional line will give a satisfactory result when this process is applied. An excellent account of the treatment of toxicological data has been given by C. I. Bliss (4-7), and the subject is also discussed, from the point of view of amount of information, in *The Design of Experiments* (2).

### *Example* 7

The following survival fractions were obtained on testing sets of eight brine-shrimps, *Artemia salina*, from a single brood, in arsenical solutions having concentrations in geometric progression.

| Solution | *C* | *D* | *E* | *F* | *G* | *H* | *I* |
|---|---|---|---|---|---|---|---|
| Total tested | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| Survived | 8 | 8 | 6 | 5 | 5 | 1 | 0 |

Since, using strengths *C*, *D* and *I*, all or none have survived, the tests with solutions *E* to *H* are available for plotting probit values, in order to obtain a provisional line. A sufficiently good line may usually be drawn graphically. A second serviceable method is to fit a straight line, without weighting, to the probits available. Thus, using Table X, we have

| Solution | Empirical Probit. |
|---|---|
| *E* | 4·33 |
| *F* | 4·68 |
| *G* | 4·68 |
| *H* | 6·15 |
| Total | 19·84 |
| Mean | 4·96 |

The slope from four equally spaced points is

$$b=\frac{3(H-E)+(G-F)}{3^2+1^2}=\cdot 546.$$

It is worth while giving some attention to obtaining a satisfactory preliminary fit, since otherwise the fitting process may have to be repeated. It is also desirable to avoid interpolation by choosing simple values. The inclusion of *C*, *D* and *I* will certainly tend to increase the slope, and may lower the average probit. For *b*, therefore, we take the value ·6, and take the probit midway between *F* and *G* to be 4·9. The provisional expected probits are those shown in the 3rd column of Table 2.

TABLE 2.

| Solution. | Independent Variate $x$. | Provisional Probit $Y$. | Working Probit $y$. | Weight $w$. | $wy$. |
|---|---|---|---|---|---|
| *C* | 3 | 2·8 | 2·4081 | ·7343 | 1·7683 |
| *D* | 4 | 3·4 | 2·9060 | 1·9002 | 5·5220 |
| *E* | 5 | 4·0 | 4·3775 | 3·5090 | 15·3606 |
| *F* | 6 | 4·6 | 4·6826 | 4·8042 | 22·4961 |
| *G* | 7 | 5·2 | 4·6776 | 5·0193 | 23·4783 |
| *H* | 8 | 5·8 | 6·0998 | 4·0208 | 24·5261 |
| *I* | 9 | 6·4 | 6·9394 | 2·4159 | 16·7649 |
|  |  |  |  | 22·4037 | 109·9163 |

The remainder of Table 2 is then filled by using Table XI.

When all die, as for solution *I*, the working probit has its maximal value, given in the 2nd column of Table XI. This is 6·9394, or ·5394 above expectation. (In the calculations for fitting, we may use either the working probits themselves to obtain the actual constants of the fitted line, or their deviations from provisional expectations to obtain adjustments of the constants provisionally adopted. Table 2 is arranged to show the first method.)

More generally, we subtract from the maximal probit a fraction of the range (3rd column of Table XI) equal to the proportion of survivors; so $G$ gives

| | | |
|---|---|---|
| Maximal working probit | 6·2759 | Range 2·5573 |
| Range × 5/8 | 1·5983 | |
| Working probit | 4·6776 | |

while for $H$ we have, similarly, 6·0998.

For $E$, if all had *lived*, we should have 3·3443, but as one-quarter have died we add one-quarter of 4·1327, giving 4·3775, while for $F$ we find, similarly, 4·6826.

In all cases the weight assigned to the evidence of each solution is the weighting coefficient, corresponding with the provisional probit, multiplied by the number of individuals tested.

From the sum of the weights, and the weighted sums of $x$ and $y$, we obtain the values of $\bar{x}$ and $\bar{y}$ according to the formulæ:

$$W = S(w), \qquad W\bar{x} = S(wx), \qquad W\bar{y} = S(wy).$$

It is convenient to calculate $S(wx^2)$ at the same time as $S(wx)$; for calculating these any suitable origin such as $F$ $(x = 6)$ may be used. The reduced sum of squares of the independent variate is then

$$A = Sw(x-\bar{x})^2 = S(wx^2) - \bar{x}S(wx).$$

With most types of machine, the values of $wy$ will be written down and summed as in Table 2; from these it is easy to obtain the reduced sum of products,

$$B = Swy(x-\bar{x}) = S(wxy) - \bar{x}S(wy).$$

These quantities are all that are needed for adjusting the regression line. For testing goodness of fit we also calculate

$$C = Sw(y-\bar{y})^2 = S(wy^2) - \bar{y}S(wy).$$

Turning to the numerical values of Table 2, we may note that the three solutions first omitted contribute nearly one quarter of the total weight, and a still higher proportion to $A$, or to the amount of information supplied as to the slope of the line. For the others the working probits differ not greatly, but appreciably, from the empirical probits first used.

Numerically, Table 2 gives

| | | | | | |
|---|---|---|---|---|---|
| $S(wx)$ | 10·7963 | $S(wx^2)$ | 60·5641 | $S(wxy)$ | 91·1157 |
| $\bar{x}$ | ·481898 | $A$ | 55·3614 | $B$ | 38·1473 |
| $\bar{y}$ | 4·906167 | | | $b$ | ·68906 |

The recalculated slope, $b = B/A$, is ·68906. This is considerably higher than the provisional value, ·6. Accepting this solution, the 50 per cent. death point is found when $Y = 5$, that is ·093833 above $\bar{y}$. We have, therefore,

$$6{\cdot}481898 + \frac{{\cdot}093833}{{\cdot}68906} = 6{\cdot}618$$

as the estimated 50 per cent. death point.

For goodness of fit we find

| | | |
|---|---|---|
| $S(wy^2)$ | 568·6512 | |
| $C$ | 29·3835 | |
| $\chi^2 = C - Bb$ | 3·0977 | $n = 5$ |

Since two constants have been fitted there are five degrees of freedom, so that there is no indication of departure from the form of the theoretical expectation. In these circumstances the variance of the estimated 50 per cent. death point may be taken to be

$$s^2 = \frac{1}{b^2}\left\{\frac{1}{W} + \frac{(x-\bar{x})^2}{A}\right\}$$

Numerically,

| | |
|---|---|
| $1/W$ | ·044635 |
| $({\cdot}136)^2/A$ | ·000334 |
| | ·044969 |
| $bs$ | ·21206 |

The standard sampling error of the estimate is therefore ·308.

This is a large-sample method, appropriate if $\bar{y}$ is sufficiently near to 5, and the slope $b$ sufficiently well determined. Failing this, the use of a standard error is not justified, and we should calculate the fiducial limits of the 50 per cent. death point, at any chosen level of significance, as Bliss has proposed (5). These are

$$\bar{x}+\frac{1}{Bb-t^2}\left\{(5-\bar{y})B\pm t\sqrt{A\left\{\frac{1}{W}(Bb-t^2)+(5-\bar{y})^2\right\}}\right\},$$

where $t$ stands for the normal deviate corresponding with the level of significance chosen. *E.g.* the limits are 5·988 and 7·294, when $t = 1{\cdot}96$.

Had $\chi^2$ been excessive, indicating causes of disturbance other than random sampling, the estimated variance would be multiplied by $\chi^2/n$, or, in the small-sample treatment, $t$ would be taken from Student's distribution for the number of degrees of freedom in $\chi^2$, and multiplied by $\chi/\sqrt{n}$ before use in the formula for fiducial limits. It should be remembered also that a significant value of $\chi^2$ may throw some doubt on the linearity of the regression.

The example has been chosen to illustrate the expeditious use of the tables. No interpolation has been employed, as it would be if the process were repeated, using the results of the first fit as a new provisional line. It is, therefore, of interest, as an indication of the precision to be expected, that a much more precise fit gives $6{\cdot}609\pm{\cdot}317$ for the 50 per cent. point, showing that this value and its precision have been very satisfactorily estimated. The slope, estimated at $\cdot6891\pm\cdot1344$, is not so good, since the more exact value is $\cdot7126\pm\cdot1574$. Here the error exceeds a tenth of the standard error of random sampling, so that, if the slope were of importance, a second fitting would have been justified. Finally, from the more exact fit we find $\chi^2 = 3{\cdot}773$ in place of 3·098, confirming the agreement with expectation, but showing also that had this been doubtful a more exact fitting might have resolved the doubt.

### *Dosage Mortality Tests involving a Natural Death Rate*

A further complication arises when there is reason to suspect that some animals have died from natural causes operating independently of the poison. If the natural mortality rate is $K$, the mortality rate, $P$, actually due to a particular dose of poison may be estimated as $p = (p^*-K)/(1-K)$, where $p^*$ is the *total* death rate observed. The probit regression line should then be fitted to the probits of $p$, not to those of $p^*$, using modified weighting coefficients obtained by multiplying the coefficients in Table XI by $P\Big/\left\{P+\frac{K}{1-K}\right\}$. Values of the modified coefficients, adapted from Finney (3a), are given in Table XI1 for values of $K$ from ·00 to ·40.

In practice, $K$ is generally unknown. If observations have been made on a control (untreated) group of animals, the mortality rate in this group, $K_c$, is an estimate of $K$; a better estimate will take account also of evidence from the lower doses of the poison, especially if one of these shows a total mortality rate less than $K_c$.

For exact work, particularly when $K$ is not well determined by the control group, the method of maximum likelihood should be applied to estimate this parameter at the same time as the coefficients $a$ and $b$ of the regression line are estimated. A provisional estimate of $K$, say $K_0$, should be used to give values of $p$ from which a provisional regression line may be constructed; $K_0$ may be taken as equal to $K_c$, but, if the number of animals in the control group is small (or even zero), a suitable guess should be based upon the values of $p^*$ for the lower doses of the poison. A table like Table 2 is prepared, with the addition of a column for the auxiliary variate $x' = Q/Z$, tabulated as a function of $Y$ in Table XI1. A weighted linear regression equation of $y$ on $x$ and $x'$ is next calculated, with the modification that the quantities $n_c(1-K_0)/K_0$ and $n_c(K_c-K_0)/K_0$ are added to the sum of squares of deviations for $x'$ and the sum of products of deviations for $x', y$ respectively. Here $n_c$ is the number of subjects in the control group, and all weighting coefficients come from the modified formula. The sum of squares of deviations for $y$, denoted by $C$, must be increased by $\frac{n_c(K_c-K_0)^2}{K_0(1-K_0)}$, as a contribution from the controls. The modified notation now used will be clear from examination of Table 2.01. The two regression coefficients are estimates of $b$ and $\delta/(1-K_0)$ respectively; $(K_0+\delta)$ is a revised estimate of $K$, and the revised regression equation is $Y = a+b(x-\bar{x})$, where $a = \bar{y}-\bar{x}'\delta/(1-K_0)$. The process may be repeated, using the values just calculated as a new provisional set if they differ seriously from the first set.

### *Example* 7.1

The first three columns of Table 2.01 give the results of a test of the toxicity of an extract of *Derris elliptica* sprayed on to the sawtoothed grain beetle, *Oryzæphilus surinamensis* (Martin (43)). The last line refers to a control group; this received the spray medium only, and in it 16.3 per cent. of the insects were affected.

In this example, $K_o$ might well be taken as equal to ·163, but, in order to simplify interpolation in Table XI1 (and to illustrate the calculations more fully), the value ·16 has been used. Values of $p$ have been calculated from each $p^*$, their empirical probits plotted against $x$, and a straight line drawn by eye on the diagram. From this line, provisional probits, $Y$, have been read. If $K_o$ were well determined, second approximations to $a$ and $b$ could be obtained by the procedure of Example 7, except that the weighting coefficients would be taken from Table XI1 by interpolation for $K = {\cdot}16$. The values of $w$, $y$, $wx$, $wy$ and the sums of squares and products from which $A$, $B$ and $C$ are derived are shown in Table 2.01.

TABLE 2.01

| Log Conc. ($x$). | No. Exposed. | No. Affected. | $p^*$. | $p$ ($K_o = {\cdot}16$). | Emp. Probit. | $Y$. | $x'$. | $y$. | $w$. | $wx$. | $wx'$ | $wy$. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2·79 | 125 | 125 | 1·000 | 1·000 | — | 7·3 | ·379 | 7·68 | 7·9 | 22·041 | 2·9941 | 60·672 |
| 2·66 | 117 | 115 | ·983 | ·980 | 7·05 | 7·0 | ·421 | 7·05 | 12·8 | 34·048 | 5·3888 | 90·240 |
| 2·49 | 127 | 114 | ·898 | ·879 | 6·17 | 6·5 | ·516 | 6·08 | 28·3 | 70·467 | 14·6028 | 172·064 |
| 2·17 | 51 | 40 | ·784 | ·743 | 5·65 | 5·6 | ·823 | 5·65 | 22·6 | 49·042 | 18·5998 | 127·690 |
| 1·57 | 132 | 34 | ·258 | ·117 | 3·81 | 3·8 | 4·557 | 3·81 | 18·5 | 29·045 | 84·3045 | 70·485 |
| — | 129 | 21 | ·1628 | — | — | — | — | — | $W = 90{\cdot}1$ | 204·643 | 125·8900 | 521·151 |

$$\bar{x} = 2{\cdot}2713,\ \bar{x}' = 1{\cdot}3972,\ \bar{y} = 5{\cdot}7841$$

$$129 \times {\cdot}84/16 = 677{\cdot}2500,\ 129 \times ({\cdot}1628 - {\cdot}16)/{\cdot}16 = 2{\cdot}258,\ 129 \times ({\cdot}1628 - {\cdot}16)^2/{\cdot}16 \times {\cdot}84 = {\cdot}01$$

| $S(wx^2)$. | $S(wxx')$. | $S(wx'^2)$. | $S(wxy)$. | $S(wx'y)$. | $S(wy^2)$. |
|---|---|---|---|---|---|
| 479·5467 | 231·7684 | 410·4217 | 1225·501 | 576·060 | 3138·30 |
| 464·8031 | 285·9324 | 175·8967 | 1183·684 | 728·165 | 3014·41 |
| $A = 14{\cdot}7436$ | $A' = -54{\cdot}1640$ | 234·5250 | $B = 41{\cdot}817$ | $-152{\cdot}105$ | 123·89 |
| | | 677·2500 | | 2·258 | ·01 |
| | | $A'' = 911{\cdot}7750$ | | $B' = -149{\cdot}847$ | $C = 123{\cdot}90$ |

If the information on $K$ is less complete, it also must be included in the maximum likelihood estimation. The additional columns $x'$ and $wx'$ are then required in Table 2.01, where $x' = Q/Z$ is taken from table XI1. From these columns, squares and products of deviations for $x'$ with $x$ and $y$ must be calculated just as were $A$, $B$ and $C$. The lower part of Table 2.01 summarises the calculations leading to equations for the regression coefficients on $x$ and $x'$. These are

$$14{\cdot}7436b - 54{\cdot}1640\left(\frac{\delta}{1-K_o}\right) = 41{\cdot}817$$

$$-54{\cdot}1640b + 911{\cdot}7750\left(\frac{\delta}{1-K_o}\right) = -149{\cdot}847.$$

The equations are best solved by the method outlined in *Statistical Methods* (1), as the inverse matrix of coefficients thus obtained leads to the variances and covariance of $b$ and $\delta$. The inverse matrix is

$$V = \begin{bmatrix} {\cdot}0867605 & {\cdot}0051540 \\ {\cdot}0051540 & {\cdot}0014029 \end{bmatrix}$$

whence

$$b = 2{\cdot}8558,\quad \delta/(1-K_o) = {\cdot}00530.$$

Multiplication of the second of these quantities by ·84 gives $\delta = {\cdot}0045$, so that the revised value for $K$, ·1645, is little different from either the provisional value or that estimated from the controls alone. Also

$$a = 5{\cdot}7841 - 1{\cdot}3972 \times {\cdot}00530 = 5{\cdot}7767.$$

The regression equation $Y = a + b(x - \bar{x})$ is easily seen to agree well with the $Y$ column of Table 2.01, so that no repetition of the calculations is necessary.

Goodness of fit may be tested by calculating

$$\chi^2 = C - Bb - B'\delta/(1-K_o) = 5{\cdot}27,$$

for which the degrees of freedom are here three (= total number of dose levels, including controls, minus 3). This gives no indication of heterogeneity. In cases of doubt, $\chi^2$ should be calculated from expected frequencies after grouping of classes with small expectations, in the manner described by Finney (3a). A large value of $\chi^2$ must be interpreted and used in the same way as for the simple probit problem (Example 7).

The 50 per cent. affected point is estimated to be

$$x_{50} = \bar{x} + (5-a)/b = 1{\cdot}999.$$

On large sample theory, the diagonal elements of $V$, $v_{11}$ and $v_{22}$, are the variances of $b$ and $\delta/(1-K_o)$ respectively, and $v_{12}$ is the covariance of these. Provided that $b^2$ is large relative to the variance of $b$, the variance of $x_{50}$ may be taken as

$$\begin{aligned} s^2 &= \frac{1}{b^2}\left\{\frac{1}{W} + v_{11}(x_{50}-\bar{x})^2 - 2v_{12}\bar{x}'(x_{50}-\bar{x}) + v_{22}\bar{x}'^2\right\} \\ &= \frac{1}{(2{\cdot}8558)^2}\left\{{\cdot}0110988 + {\cdot}0867605 \times ({\cdot}2720)^2 \right. \\ &\qquad \left. + 2 \times {\cdot}0051540 \times 1{\cdot}3972 \times {\cdot}2720 + {\cdot}0014029 \times (1{\cdot}3972)^2\right\} \\ &= {\cdot}002964. \end{aligned}$$

The standard error of the 50 per cent. point is therefore ·0544. Here $(1{\cdot}960)^2 v_{11}/b^2 = {\cdot}041$, a value small enough to justify the use of $s$ in assigning 95 per cent. fiducial limits to $x_{50}$; if this quantity exceeds ·05, the limits ought to be calculated by an extension of the procedure illustrated for Example 7. The variance of $\delta$ is $v_{22}(1-K_o)^2$, which is also therefore the variance of the revised estimate of the natural response rate; hence this rate is estimated to be ${\cdot}1645 \pm {\cdot}0314$.

### *Other Transformations*

The normal transformation may be looked upon as one of a series of methods of transforming percentages into more suitable variables. Of these the transformation of a probability varying from 0 to 1 into an angle varying from 0° to 90°, according to the relationship

$$p = \sin^2 \phi$$

is remarkable in that the amount of information concerning $\phi$ is constant. In consequence, in large samples, the sampling variation tends to normality with a variance which depends only on the number of observations on which the percentage is based. The weighting coefficient for this transformation is constant. In other respects Tables XII to XIV afford the same facilities for handling percentages transformed in this way as do Tables IX to XI for the normal transformation. This transformation is equivalent to replacing the normal curve by one of which the ordinate is $\sin 2\phi$.

A third transformation of some interest, which is readily available from these tables, is given by

$$z = \tfrac{1}{2}(\log_e p - \log_e q)$$

or

$$p = e^{2z}/(1+e^{2z}).$$

If $r = 2p-1$, the values of $z$ are given in Table VII. With this transformation we may use

| | |
|---|---|
| Maximal working $z$ . . . . . | $Z + \frac{1}{2P} = Z + \frac{1}{1+R}$ |
| Range . . . . . . . . | $\frac{1}{2PQ} = \frac{2}{1-R^2}$ |
| Weighting coefficient . . . . | $4PQ = 1-R^2$ |

where $Z$ is the expected value of $z$ given by any provisional fitting, and the value of $R$ is taken from that of $Z$, using Table VII. This transformation is equivalent to replacing the normal curve by one of which the ordinate is $\frac{1}{2}\,\mathrm{sech}^2 z$; it has been called the "logit" transformation.

Intermediate in character between the distributions of the probit and the logit is the deviate appropriate to gene frequencies determined by selection and diffusion, where $p$, $q$ are related to the standard deviate $x$ by the equation

$$\frac{d^2q}{dx^2} = 4pqx.$$

The name "legit" has been proposed for this transformation. The necessary tables have been given (Fisher, 56), using the general maximum likelihood values of the working deviate for given provisional values, namely

| | |
|---|---|
| Maximum . . . . . . . . . | $X+Q\frac{dX}{dP}$ |
| Minimum . . . . . . . . . | $X-P\frac{dX}{dP}$ |
| Weighting coefficient . . . . . . . | $\frac{1}{PQ}\left(\frac{dP}{dX}\right)^2$ |

where $P$, $Q$ stand for the probability corresponding to the provisional value $X$.

In addition to their toxicological and similar applications, transformations are of use in the analysis of material which has a very skew distribution, for the purpose of equalising the variance of quantities of widely different magnitudes which it is desired to include in the same analysis, and for the purpose of reducing the observations to a scale on which the effects of associated variables are more nearly constant. Thus when it is required to perform an analysis of variance on a set of percentages, *e.g.* of the deaths of seedlings obtained from plots of a replicated experiment, it is frequently advantageous to transform these percentages by means of the angular transformation. As we have seen, this has the effect, in large samples, of equalising the component of variance due to sampling, when each percentage is based on the same number of observations. The square root transformation

$$y=\sqrt{x}$$

has the same property when the observations on each plot are distributed according to a Poisson distribution. This transformation is equivalent to the angular transformation at each end of the percentage scale. (In this connection see W. G. Cochran (8).)

The logarithmic transformation

$$y=\log x$$

is useful when dealing with material on which the effects of other variables may be expected to be proportional to the numbers involved. The influence of a given increase in temperature on the number of insects caught in a light trap, for example, is likely to be proportional to the number of active insects, in which case the increment in the logarithm of the catch will be the same whatever the number of insects, unless some of the numbers are very small. In this connection it may be noted that the transformation $y=\log(x+1)$ is sometimes convenient, being substantially equivalent to $y=\frac{1}{3}\sqrt{x}$ when $x$ is less than 10 and to $y=\log x$ for larger numbers.

A further use of transformations is for the purpose of facilitating interpolation by obtaining a transformed function which is much more nearly linear. One example of this, the use of the reciprocals of $n_1$ and $n_2$ in interpolation in the table of $z$ (Table V), has already been mentioned. The logarithmic transformation and the transformation of Table VII have frequently proved of use when dealing with functions whose argument is a probability.

## TABLE XIV1. SCORES FOR LINKAGE DATA FROM INTERCROSSES

Much linkage data with plants is obtained by selfing double heterozygotes, which may be in coupling or in repulsion. In some cases, especially with animals, single or double backcrosses may also be available. The compilation of such complex data, for the estimation of linkage, or the examination of heterogeneity, is much facilitated by efficient scores, based on the method of maximum likelihood, such as have been developed for human genetics.

Such scores possess the properties (i) that the scores are linear functions of the frequencies observed, (ii) that the expected score is zero if the trial value is correct, (iii) that the ratio of change of the expected score with change of trial value is equal to its sampling variance, and to the amount of information to be expected from such a body of data as is available. Estimation from a single body of data thus consists in finding the value for which the total score vanishes. The efficiently weighted combination of different lots of data consists simply in adding their scores. Homogeneity is tested by recognising that the square of the score divided by the amount of information is $\chi^2$ for one degree of freedom. If $m$ is the expectation of any distinguishable class, expressed as a function of the unknown parameter $\theta$, the efficient score for cases observed in that class may be taken to be

$$\frac{1}{m}\frac{\partial m}{\partial\theta}.$$

For double heterozygotes intercrossed the two singly recessive classes have the same expectation and the same score. Table XIV1 gives the means of scoring such data in coupling and repulsion for each 1 per cent. of the recombination fraction, $y$, the quantities tabulated being :

| | Coupling | Repulsion |
|---|---|---|
| Double dominant | $\frac{2(1-y)}{2+(1-y)^2}$ | $-\frac{2y}{2+y^2}$ |
| Single recessives | $\frac{1}{y}-\frac{1}{2-y}$ | $\frac{1}{1-y}-\frac{1}{1+y}$ |
| Double recessive | $\frac{2}{1-y}$ | $-\frac{2}{y}$ |

The signs in the table are for $\theta = 1-y$, where $y$ is the recombination fraction, in order that evidence for linkage ($y<\cdot5$) may be supplied by positive scores at 50 per cent.

Data should be scored at two adjacent values giving positive and negative scores. One hundred times the difference then gives the amount of information to a good approximation. For more accurate work, and especially for close linkage, four adjacent values with four-point interpolation may be used. The four-point value for the actual amount of information obtained at the value $x+\theta$ is

$$(u_{x+1}-u_x)-\frac{1}{6}(1-3\theta^2)\delta^2u_{x+1}+\frac{1}{6}(1-3\phi^2)\delta^2u_x,$$

where $\theta+\phi = 1$.

Since the Third Edition, similar tables have been calculated independently by D. J. Finney (57), who gives a good account of the origin and use of this mode of application of the method of maximum likelihood.

### *Example* 7.2

For two factors in the Sweet Pea we have the data :

| | Double dominant | Single recessives | Double recessive |
|---|---|---|---|
| Coupling | 268 | 30 | 79 |
| Repulsion | 324 | 344 | 0 |

Scoring these at 6 per cent., 7 per cent., 8 per cent. and 9 per cent., we find

| | 6 per cent. | | 7 per cent. | | 8 per cent. | | 9 per cent. |
|---|---|---|---|---|---|---|---|
| Score | $-119\cdot701$ | | $-43\cdot367$ | | $15\cdot165$ | | $61\cdot852$ |
| $\delta^1$ | | $+76\cdot334$ | | $+58\cdot532$ | | $+46\cdot687$ | |
| $\delta^2$ | | | $-17\cdot802$ | | $-11\cdot845$ | | |

To find the score at an intermediate value, such as 7·811 per cent., we find the inner interpolate, 4·102, and, since the linkage is close, the four-point value, 5·191. A method of calculating this is given in the section on *Interpolation* at the end of the introduction. The first approximation to the amount of information obtained is 5853·2, and the adjustment is −457·0, giving 5396·2 at this value.

## TABLE XIV2. SEGMENTAL FUNCTIONS, $\alpha$, $\beta$, $\gamma$, $\delta$

These functions are defined for all values of their argument as the sums of sub-series of the exponential series, containing every fourth term. The four sub-series having leading terms of degrees 0, 1, 2, and 3, define the functions $\alpha$, $\beta$, $\gamma$, and $\delta$.

They are closely connected with the hyperbolic and circular functions by the identities

$$\alpha+\gamma = \cosh, \qquad \beta+\delta = \sinh,$$
$$\alpha-\gamma = \cos, \qquad \beta-\delta = \sin.$$

Two elementary identities are found by eliminating the argument.

$$\alpha^2+\gamma^2-2\beta\delta = 1$$
$$\beta^2+\delta^2-2\alpha\gamma = 0.$$

Differentiation with respect to the argument merely permutes the symbols in the order ($\delta\gamma\beta\alpha$).

The functions are peculiarly adapted to specifying the frequencies of non-recombinant and recombinant gametes of all types in terms of the metrical positions of the centromere, a series of any number of markers, and the terminus of the chromosome arm. The argument of each function is twice the metrical length of the segment. Thus with four markers and five segments the frequency of gametes non-recombinant in every segment is given by a simple combinatorial rule to be the sum of the following sixteen products,

$$\begin{array}{cccc}
\alpha\alpha\alpha\alpha\alpha & \alpha\alpha\alpha\beta\delta & \beta\alpha\delta\alpha\alpha & \beta\alpha\alpha\alpha\delta \\
\alpha\beta\alpha\delta\alpha & \alpha\alpha\beta\alpha\delta & \beta\alpha\alpha\delta\alpha & \beta\alpha\delta\beta\delta \\
\alpha\alpha\beta\delta\alpha & \alpha\beta\alpha\alpha\delta & \beta\delta\alpha\alpha\alpha & \beta\delta\beta\alpha\delta \\
\alpha\beta\delta\alpha\alpha & \alpha\beta\delta\beta\delta & \beta\delta\beta\delta\alpha & \beta\delta\alpha\beta\delta
\end{array}$$

in which, for the segment next the centromere the appropriate function is either $\alpha$ or $\beta$, while for the segment ending at the terminus it is either $\alpha$ or $\delta$. Intervening segments may have $\alpha$, $\beta$ or $\delta$, provided that $\beta$ and $\delta$ appear in equal numbers and alternate.

The corresponding expressions for gametes having recombination in one or more of the five segments are found simply by making the substitution $(\alpha\gamma)(\beta\delta)$ in each corresponding column. In particular, if as is usual, there is no marker at the centromere, or at the terminus, the eight observable classes in a four-point test will be specified by permutations in the three middle columns of the expression :—

$$(\alpha+\gamma)\left\{\begin{matrix}\alpha\alpha\alpha\\ \beta\alpha\delta\\ \alpha\beta\delta\\ \beta\delta\alpha\end{matrix}\right\}(\alpha+\gamma) \qquad (\beta+\delta)\left\{\begin{matrix}\alpha\delta\alpha\\ \alpha\alpha\delta\\ \delta\alpha\alpha\\ \delta\beta\delta\end{matrix}\right\}(\alpha+\gamma)$$

$$(\alpha+\gamma)\left\{\begin{matrix}\alpha\alpha\beta\\ \alpha\beta\alpha\\ \beta\alpha\alpha\\ \beta\delta\beta\end{matrix}\right\}(\beta+\delta) \qquad (\beta+\delta)\left\{\begin{matrix}\alpha\alpha\alpha\\ \alpha\delta\beta\\ \delta\beta\alpha\\ \delta\alpha\beta\end{matrix}\right\}(\beta+\delta).$$

With the aid of the tables these eight frequencies may be easily calculated for any assigned values of the five metrical lengths of the segments. Thus, starting from the left, we may calculate

$$P = (\beta+\delta)\alpha+(\alpha+\gamma)\beta,$$
$$Q = (\beta+\delta)\delta+(\alpha+\gamma)\alpha,$$

beside which we can calculate the parallel quantities,

$$P' = (\beta+\delta)\gamma+(\alpha+\gamma)\delta,$$
$$Q' = (\beta+\delta)\beta+(\alpha+\gamma)\gamma,$$

to be used in the cases in which recombination takes place in the second segment.

To bring in the third segment we make the similar calculation

$$R = P\alpha+Q\beta, \qquad R' = P\gamma+Q\delta,$$
$$S = P\delta+Q\alpha, \qquad S' = P\beta+Q\gamma,$$

with the same variation for recombination in the third segment, and, of course, using $P'$, $Q'$ in place of $P$ and $Q$ for the cases involving recombination in the second segment.

Equally, for the fourth segment, we take for each of the four pairs of values $R$, $S$

$$T = R\alpha+S\beta, \qquad T' = R\gamma+S\delta,$$
$$U = R\delta+S\alpha, \qquad U' = R\beta+S\gamma,$$

giving eight pairs of values for the eight distinguishable classes of gamete. Using the segmental functions of the fifth segment, we have from each of these pairs

$$T(\beta+\delta)+U(\alpha+\gamma)$$

the eight values of which should check to a total calculable in advance, namely the hyperbolic cosine of twice the total arm-length from centromere to terminus.

The expectations corresponding with assigned lengths of segments are then found by dividing each by $\cosh 2U$, where $U$ now stands for the total metrical arm length. (References 58, 59, 60, 61).

### *Example* 7.3

An arm is divided by four markers $P$, $Q$, $R$ and $S$ into five segments with metrical lengths 10, 25, 20, 15 and 5 $cM$, where the centimorgan is defined as 1 per cent. of the metrical unit. To apply the computational procedure outlined above, these lengths are first doubled to give the first two decimal places of the arguments of the segmental functions to be used. The first argument is therefore ·20 and the second ·50; we take then directly from the table :—

$$(\beta+\delta)(\cdot 20) = \cdot 2013360 \qquad \alpha(\cdot 50) = 1\cdot 0026043 \qquad \delta(\cdot 50) = \cdot 0208349$$
$$(\alpha+\gamma)(\cdot 20) = 1\cdot 0200668 \qquad \beta(\cdot 50) = \cdot 5002604 \qquad \gamma(\cdot 50) = \cdot 1250217$$

from which the pairs $P$, $Q$ and $P'$, $Q'$ are calculated directly, as shown in Table 2.02 with the remaining calculations completed.

TABLE 2.02

| | $R$, $S$. | $T$, $U$. | | Frequency per cent. | Segments recombining. |
|---|---|---|---|---|---|
| | | 1·4348573 | 1·1899512 | 50·584 | None |
| | 1·1237739 | 1·0410167 | | | |
| | 1·0356102 | | | | |
| | | ·0552312 | ·3912112 | 16·630 | (4) |
| $P$ ·7121594 | | ·3837585 | | | |
| $Q$ 1·0269182 | | | | | |
| | | ·1780864 | ·3871909 | 16·459 | (3) |
| | ·0679309 | ·3675134 | | | |
| | ·3670838 | | | | |
| | | ·0047088 | ·0375561 | 1·596 | (34) |
| | | ·0368998 | | | |
| | | ·2065418 | ·2515250 | 10·692 | (2) |
| | ·1377938 | ·2296870 | | | |
| | ·2289896 | | | | |
| | | ·0072313 | ·0526285 | 2·237 | (24) |
| $P'$ ·0464244 | | ·0516457 | | | |
| $Q'$ ·2282509 | | | | | |
| | | ·0172024 | ·0387828 | 1·649 | (23) |
| | ·0061490 | ·0368752 | | | |
| | ·0368351 | | | | |
| | | ·0004425 | ·0035643 | ·152 | (234) |
| | | ·0035024 | | | |
| | | | ——— | ——— | |
| | | | 2·3524100 | 99·999 | |
| | | cosh 1·50 | 2·3524096 | | |

Since the differential coefficients of these functions are the same functions permuted, the rates of change of the eight frequencies for variations of the segment lengths are obtained by calculations involving the same tabulated values. Segment lengths may then be estimated from observed frequencies by the method of maximum likelihood. Estimates of unmarked segments will, naturally, have a very low precision.

## TABLES XV AND XVI. LATIN SQUARES

Latin squares of size 7×7 and smaller have been enumerated (11), (22), (66). For 4×4 and 5×5 squares all the standard squares (*i.e.* squares with first row and first column in the standard order) are shown in Table XV, except that only one of each pair of conjugate 5×5 squares is given (two standard squares are conjugate if the rows of one are the columns of the other). From each 4×4 standard square 4 ! 3 ! —1 non-standard squares, all different, may be generated by permuting all the rows except the first, and all columns. Thus each 4×4 standard square represents a set of 144 squares. There are consequently 4×144 or 576 4×4 Latin squares. Similarly, there are 56×2880 or 161,280 5×5 Latin squares. There is one 2×2 standard square and one 3×3 standard square, giving two 2×2 squares and twelve 3×3 squares in all.

For 6×6 squares one square of each transformation set (*i.e.* set generated from one of its members by permutation of rows, columns, and letters), or of each pair of conjugate transformation sets, is shown. Such permutation of rows, columns and letters will generate $(6\,!)^3$ or 373,248,000 squares, of which 6×6 ! or 4320 are standard squares. These squares are, however, not necessarily all different, the largest number actually obtained in any one $6 \times 6$ set being one-quarter of the possible number, *i.e.* 1080 standard squares, and the smallest number one 216th, *i.e.* 20 standard squares. The actual numbers in each transformation set are shown in the table. There are in all 9408 standard $6 \times 6$ squares, giving 812,851,200 $6 \times 6$ squares in all.

By permuting with each other the three categories, rows, columns, and letters, six sets are formed, not necessarily all different. These sets are said to be *adjugate*. In the case of the $6 \times 6$ squares each pair of conjugate sets and the set immediately following them in the table form an adjugate trio of sets. The remaining sets are self-adjugate.

Norton (22) has enumerated 562 sets of $7 \times 7$ squares, of 146 species, containing 16,927,968 standard squares. Later, A. Sade, using a different method of enumeration, obtained the value 16,942,080 for the number of standard $7 \times 7$ squares; he has subsequently discovered a 147th species (66) which accounts for the discrepancy of 14,112 standard squares. Four examples are shown in the table. Square (*a*) is a member of a transformation set in which the squares generated by the permutation of rows, columns and letters are all different. It thus contains $(7\,!)^3$ or 128,024,064,000 squares, of which $7 \times 7\,!$ or 35,280 are standard squares. The sets generated by the adjugate permutations are also all different. The example thus stands for an aggregate of 211,680 standard squares. No Graeco-Latin squares can be formed with any of these.

Square (*b*) differs from square (*a*) only by the interchange of two pairs of letters, *B* and *C* in the 5th and 6th columns. It belongs to a set of only 11,760 standard squares, which is one of an adjugate trio of sets, giving 35,280 standard squares in all. One-seventh (1680) of the standard squares of one of these sets are self-conjugate. Again no Graeco-Latin square can be formed.

The third illustration is a Graeco-Latin square, with the fourth category (Greek letters) represented by numeral suffixes. No fifth category can be added. The suffixes are standardised by assigning them in their natural order to the first row. The Latin square involved is one of a set of only 210 standard Latin squares, but the same transformation of the Graeco-Latin square yields a set of 1680 standard Graeco-Latin squares, containing eight different standard arrangements of the suffixes in each Latin square. These eight solutions are in fact all that exist, so that this set of Latin squares belongs to only one Graeco-Latin set. Permutation of the three categories, rows, columns, and letters, yields the same set, but if suffixes are interchanged with rows, columns, or letters, three more sets of 1680 standard Graeco-Latin squares are formed (6720 in all).

The three new sets all involve the cyclic Latin square shown as the fourth example. There are only 120 standard Latin squares in this set, so that each has 14 solutions in each of the Graeco-Latin sets, or 42 in all. Under orthogonal squares are given six $7 \times 7$ squares, of which the first is the above Latin square, while the rest are five more Graeco solutions which belong to a second set and are mutually orthogonal.

Little is known about squares larger than $7 \times 7$, other than the orthogonal properties of the special sets shown in Table XVI. In Table XV a single example of each size from $8 \times 8$ to $12 \times 12$ is given. These examples will suffice (after permutation of rows, columns, and letters at random) for experimental arrangements requiring Latin squares of these sizes.

To select a square at random from all possible squares of a given size up to $6 \times 6$ proceed as follows:

(1) Select one of the given squares by using the key numbers printed below each square, selecting a number at random from all possible key numbers. If the key number falls in the second of two groups use the conjugate square.

(2) In the case of $3 \times 3$, $4 \times 4$, and $5 \times 5$ squares permute all rows except the first of the selected square, and all columns. Alternatively permute all rows except the first and assign the letters to the treatments at random. For $6 \times 6$ squares permute all rows and all columns at random, and then assign the letters to the treatments at random.

Table XVI gives complete sets of orthogonal Latin squares of sides 3, 4, 5, 7, 8, and 9. Two Latin squares are orthogonal to each other if, when they are superimposed, every letter of one square occurs once and once only with every letter of the other. Such a pair of squares (one square being written with Greek and the other with Latin letters) form a Graeco-Latin square.

If a complete set of $p-1$ orthogonal squares of side $p$ exists, the $p^2-1$ degrees of freedom corresponding to the contrasts of the $p^2$ cells of the square split up into $p+1$ sets of $p-1$ degrees of freedom, corresponding to the contrasts between the rows, between the columns, and between the letters of each of the $p-1$ squares. A set of less than $p-1$ orthogonal squares gives a similar but incomplete partition.

Complete sets of orthogonal squares are known to exist for all prime numbers and powers of primes. (See *The Design of Experiments*.) The rule of formation for all prime numbers will be apparent from inspection of the sets for $p = 5$ and $p = 7$, each row being formed by moving the previous row 1, 2, 3 . . . places to the left. The existence of complete sets of orthogonal squares, for all prime-power values of $p$, was proved independently by R. C. Bose (63) and W. L. Stevens (64). See also R. C. Bose and K. R. Nair (65).

The rows of the $8 \times 8$ squares may be taken to represent the 8 combinations of three two-level factors *A*, *B*, and *C*, in the order (1), *a*, *b*, *ab*, *c*, *ac*, *bc*, *abc*, and the columns may be taken to represent the similar combinations of a further three factors *D*, *E*, and *F*. The 64 cells of each square will then represent the 64 combinations of the 6 factors. The numbers of each square will divide these combinations into 8 groups of 8, the contrasts among which will represent a set of 7 degrees of freedom orthogonal to $A$, $B$ and $C$ and their interactions, and to $D$, $E$, and $F$ and their interactions, and also to the six other sets of seven degrees of freedom given by the other six squares. In the squares shown these degrees of freedom correspond to specific interactions between the six factors. The interactions corresponding to 3 degrees of freedom from each set are shown in the table, the remainder being given by the " generalised " interactions of these three. Note the law of formation of each set from the preceding one, corresponding to the cyclic interchange of all the rows of the square except the first.

The set of 9×9 squares possesses similar properties. The actual degrees of freedom confounded, when the cells of the squares represent the 81 combinations of four factors at three levels, are given in terms of the notation developed by Yates (3).

Besides their use in ordinary factorial design, complete sets of orthogonal squares serve as the basis of quasi-factorial and quasi-Latin square designs (balanced lattices and lattice squares) and also serve to generate other designs in balanced incomplete blocks. (See (15), (16), (17), and Tables XVII-XIX.) Pairs and larger incomplete sets of orthogonal squares (Græco- and hyper-Græco-Latin squares) also provide designs for varietal trials superseding the so-called semi-Latin squares (3).

An orthogonal pair of squares exists for all odd numbers, being formed of two " diagonal " squares (squares of the type of the 5×5 squares I and IV). Such a pair also exists for all numbers which are multiples of 4. The process of generation, which depends on the existence of orthogonal pairs of squares for factors of the number, may be illustrated for a pair of 12×12 squares. Let $\alpha$ and $\alpha'$ represent the 4×4 squares I and II, $\beta$ and $\beta'$, and $\gamma$ and $\gamma'$, the same squares with 1, 2, 3, 4 replaced by 5, 6, 7, 8 and 9, 10, 11, 12 respectively. Then

$$\begin{array}{ccc} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{array} \quad \text{and} \quad \begin{array}{ccc} \alpha' & \beta' & \gamma' \\ \gamma' & \alpha' & \beta' \\ \beta' & \gamma' & \alpha' \end{array}$$

form an orthogonal pair of 12×12 squares. The process will generate larger orthogonal sets if such sets exist for both factors of the number. Thus there will be a set of three squares of side 20.

It has been shown that no orthogonal pair of 6×6 squares exists, but larger even numbers not divisible by 4 have not been investigated.

## TABLES XVII TO XIX. BALANCED INCOMPLETE BLOCKS

If $v$ varieties (or treatments) are to be compared in randomized blocks of $k$ experimental units ($k$ being less than $v$), block differences can be simply eliminated if the arrangement is such that every two varieties occur together in the same number ($\lambda$) of blocks. If $r$ is the number of replicates, and $b$ the number of blocks, the number of experimental units is $rv(= bk)$, and $\lambda = r(k-1)/(v-1)$. Examples of experimental material and processes in which this type of arrangement, called *balanced incomplete blocks*, is likely to be of use are : litters of animals, monozygotic twins and other paired material, laboratory and technological processes in which the nature of the process or the apparatus available imposes a definite limit on the number of treatments in a group, agricultural experiments on commercial farms when each farm is only prepared to undertake a small number of treatments, and varietal trials involving a large number of varieties. (See also Yates (16).)

Arrangements satisfying the required conditions are clearly provided by taking all possible combinations of the $v$ varieties, $k$ at a time, but if $v$ is at all large, the number of replicates required will be very large. Arrangements involving fewer replicates are therefore of importance. Tables XVIII and XIX give indexes, by number of replications ($r$), and by number of units in a block ($k$) respectively, of the arrangements requiring ten or less replications which are known to exist, and of all arithmetically possible arrangements, the existence of which had not been disproved at the date of the first edition. Since then the non-existence of solutions of Nos. 10 and 14 has been demonstrated by Q. M. Hussain (44, 45), and No. 8 has been eliminated by H. K. Nandi (46). These cases are marked by an asterisk. Cases not yet solved are marked by a dagger. General discussions of the construction of incomplete block designs have been given by R. C. Bose (31), and C. R. Rao (47). The solutions for Nos. 26 and 27 and those for Nos. 17 and 20 are due to K. N. Bhattacharya (50, 32). Schützenberger (48) has since proved that if $b = v$, and they are even, then a solution is impossible unless $r-\lambda$ is a perfect square. This confirms Hussain's rejection of No. 10, and excludes No. 30. Theorems of S. Chowla and H. J. Ryser (53) and S. S. Shrikhande (49) confirm the impossibility of No. 14, and exclude other cases (beyond the range of the table) with $b = v$, when they are odd. W. S. Connor (51) has found that designs Nos. 12 and 28 are impossible.

The combinatorial solutions which are referred to by numbers are shown in Table XVII. Those involving all combinations are indicated by *u* (unreduced), and the two series derivable from sets of orthogonal squares (described below) by *o.s.* When $k > \frac{1}{2}v$ arrangements can be obtained from arrangements for the same number of varieties in blocks of $v-k$ units, by replacing each block by its complement, *i.e.* by a block containing all the varieties missing from the original block.

Sometimes the blocks may be arranged in groups so as to minimise the inter-block variance. Each such group must contain an equal number of replicates of each variety. In general it is not possible so to arrange things that each group of blocks contains only a single replication. Such groups as are possible in the designs of Table XVII are indicated in the table itself. Other arrangements possessing this property are possible in certain cases when a multiple of the number of replications required for the designs of the Table are used. Thus when $k = \frac{1}{2}v$ a balanced design and its complement will give such a grouping. Within each group the arrangement of the blocks should be at random.

Some of the combinatorial solutions are derivable very simply by cyclic substitutions. In such cases one block only for each family is given in Table XVII. In solution number 2, for example, the block *aefgi* is given. The contents of the other 10 blocks are obtained progressively by replacing each letter by the next in order, with the convention that the last letter *k* is followed by *a*. Thus the blocks are *aefgi*, *bfghj*, *cghik*, *dhija*, etc.

For the dicyclic solutions a similar procedure holds. In solution number 5, for example, the 16 varieties are represented by 4 letters in combination with 4 suffixes. The 16 blocks are generated by performing the cyclic substitution on the letters (thus obtaining 4 blocks) and then performing the cyclic substitution on the suffixes of each of these 4 blocks.

Some of the cyclic and dicyclic solutions consist of more than one family, each family being generated by a cyclic or dicyclic substitution of the above type.

Two series of arrangements and their complements are derivable from complete sets of orthogonal Latin squares (Table XVI). In the first, for $p^2$ varieties in $p(p+1)$ blocks of $p$ plots ($r = p+1$, $\lambda = 1$), each of the $p^2$ cells of the set of squares is taken to represent a variety, the blocks being determined by the rows, the columns and the numbers of the $p-1$ squares. In the second, for $p^2+p+1$ varieties in $p^2+p+1$ blocks of $p+1$ plots ($r = p+1$, $\lambda = 1$), $p^2$ of the varieties are divided as in the first arrangement, each of the $p+1$ additional varieties being allotted to one of the $p+1$ sets of blocks so formed, while the last block consists entirely of these additional varieties. Thus the set of 4×4 squares gives the following arrangements for 16 and 21 varieties (the latter including the bracketed varieties) :

| | | | | | |
|---|---|---|---|---|---|
| *abcd* (*q*) | *aeim* (*r*) | *afkp* (*s*) | *agln* (*t*) | *ahjo* (*u*) | (*qrstu*) |
| *efgh* (*q*) | *bfjn* (*r*) | *belo* (*s*) | *bhkm* (*t*) | *bgip* (*u*) | |
| *ijkl* (*q*) | *cgko* (*r*) | *chin* (*s*) | *cejp* (*t*) | *cflm* (*u*) | |
| *mnop* (*q*) | *dhlp* (*r*) | *dgjm* (*s*) | *dfio* (*t*) | *dekn* (*u*) | |

The second of these series, at least up to 10 replications, may also be generated by the cyclic substitutions shown in Table 2.1. For convenience the varieties are here represented by numbers instead of letters. The first series may then be obtained from the second by deleting any set of varieties occurring in the same block.

TABLE 2.1. CYCLIC SOLUTIONS FOR BALANCED INCOMPLETE BLOCKS

| Problem. | Modulus ($v$). | Selected numbers. |
|---|---|---|
| *o. s.* 2 | 7 | 1, 2, 4 |
| *o. s.* 3 | 13 | 1, 2, 4, 10 |
| 2 | 11 | 1, 5, 6, 7, 9 |
| *o. s.* 4 | 21 | 1, 4, 5, 10, 12 |
| *o. s.* 5 | 31 | 1, 2, 4, 11, 15, 27 |
| 9 | 15 | 1, 2, 3, 5, 6, 9, 11 |
| *o. s.* 7 | 57 | 1, 4, 6, 14, 15, 21, 33, 37 |
| 19 | 19 | 1, 3, 5, 6, 7, 8, 11, 14, 15 |
| 23 | 37 | 1, 2, 4, 8, 18, 25, 26, 30, 36 |
| *o. s.* 8 | 73 | 1, 2, 4, 8, 16, 32, 37, 55, 64 |
| *o. s.* 9 | 91 | 1, 2, 7, 11, 24, 27, 35, 42, 54, 56 |

To these may be added the unreduced ($u$) designs with $k = v-1$. See also the dicyclic solutions in one family (No. 5), in two families (No. 13), and the cyclic solutions in two families (Nos. 4, 11, 29) and in three families (No. 18) of Table XVII. The unreduced ($u$) design with $k = 2$ and $v$ odd has an obvious cyclic solution in $\frac{1}{2}(v-1)$ families, and by taking complements a corresponding result is obtained when $k = v-2$.

### *Youden's Squares*

It has been shown (Youden, 21 ; Smith and Hartley, 52) that when $b = v$ it is possible so to arrange the order of the varieties in the blocks that each variety occurs once in each position in the block, and it is thus possible to eliminate experimental differences between these positions, each of which comprises a complete replication.

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For example, experimental material consisting of 13 plants each with 4 corresponding leaves may be assigned to 13 treatments so that each treatment occurs once on each type of leaf. At the same time the 13 plants constitute randomized blocks of 4 treatments each, in which each treatment occurs on the same plant once with every other treatment. Similarly a varietal trial of 13 varieties in blocks of 4 may be arranged on the ground in a 4×13 rectangle of plots so that both row and column differences are eliminated from the varietal comparisons.

In drawing up an actual experimental arrangement of this type the blocks should be assigned to the plants (or arranged on the ground) at random, and the experimental treatments (or varieties) should be similarly assigned to the letters.

A cyclic or dicyclic substitution which gives an incomplete block solution will also give the corresponding Youden square, provided that each block is written in the order in which it is generated. The complementary Youden square solution may be obtained by performing the same cyclic substitution on the block which is complementary to the given block. Of the remaining solutions of Table XVII for which $b = v$, No. 20 is arranged in the form of a Youden square, and a Youden square form of the solution for No. 27 is shown in Table 2.2.

TABLE 2.2. YOUDEN SQUARE SOLUTION ($b = v = 31$, $r = k = 10$)

*a b c d e f g h i j*
*b l q u p a t y e o*
*c k d y x o i t s v*
*d q m w A c D B y a*
*e B x b m E d n t C*
*f s C o c m e u z D*
*g D E a n l o m k i*
*h c y z C i n E q p*
*i m t g u j B q v z*
*j y h f t n m o w r*
*k o b s h w q C B g*
*l C r t a g w z c x*
*m z f p g y A x b k*
*n A s C y u j g d l*
*o n p r v B c e g A*
*p d o l z h x D j B*
*q r k n j D b c x u*
*r f l i B C v b D y*
*s t n D b v z A a h*
*t p i k D e C j A w*
*u h B c E A k l f t*
*v j w E l b s p m c*
*w u z A i d E r o b*
*x v A j o q a f C E*
*y e j B k z r a E s*
*z w v e q k f d l n*
*A x e m r s l i h q*
*B i a x s p u w n f*
*C a u v d r h k p m*
*D E g q f t p s r d*
*E g D h w x y v u e*

Cyclic and dicyclic solutions in two or more families provide an extension of Youden squares. In these the differences corresponding to the positions in the blocks of each family may be eliminated, *e.g.* 9 degrees of freedom in solution No. 29 for 10 replicates of 41 varieties. Solution 15 is also given in the form of an extended Youden design. Each variety occurs 3 times in each position in the blocks, and the 2 degrees of freedom between the three positions can be eliminated. This form of No. 15 (and Youden solutions of Nos. 20 and 27) was given by S. S. Shrikhande (54) in a study of designs for two-way elimination of heterogeneity.

Besides balanced incomplete blocks and Youden squares a number of different types of non-balanced arrangement are also available, namely quasi-factorial arrangements, described in (25) and (26), lattice squares (27), and partially balanced arrangements (28). The various types of arrangement are reviewed in (3) and (29). The experimenter should acquaint himself with their advantages and disadvantages before deciding which is most appropriate for his own needs.

### *Analysis of Balanced Incomplete Block Experiments*

In the analysis of incomplete block experiments the block differences may be eliminated entirely from the varietal comparisons (Yates, 16), or, as has been shown later (Yates, 23), the information contributed by these comparisons may be combined with that provided by the intra-block comparisons, with weights depending on the relative accuracy of the two sets of comparisons (as estimated from the experimental results).

If the choice of blocks has been successful in substantially reducing the variability of the material, then the amount of information contributed by the inter-block comparisons will be small, but in the limiting case when the inter-block and intra-block comparisons are of equal accuracy, the fraction of the total information contained in the inter-block comparisons is given by the efficiency factor $E = \frac{1-1/k}{1-1/v} = \frac{v\lambda}{rk}$, which approaches the value $\frac{1}{2}$ when $k = 2$ and $v$ is large.

In general, therefore, it is worth recovering the inter-block information, except in small experiments with less than 10 degrees of freedom for blocks, in which the relative weights cannot be determined with sufficient accuracy. If the method described below is followed the amount of additional computation is not great.

If $V_s$ be the sum of all the yields of variety $s$, $T_s$ the sum of all the block totals of blocks containing variety

$s$, $T'_s$ the sum of all the remaining block totals, and $G$ the total yield of all plots, the estimates of the varietal differences derived from the intra-block comparisons are obtained from the quantities

$$Q_s = V_s - T_s/k,$$

or, as is more convenient when $k > \frac{1}{2}v$,

$$Q'_s = V_s + T'_s/k.$$

The actual differences in units of the total yield of the $r$ replicates are given by the differences of

$$Q_s/E \quad \text{or} \quad Q'_s/E,$$

the sum of the first set being zero, and of the second set $rG/\lambda v$.

The error variance of these latter sets of quantities is $r/Ew$, where $1/w$ is the intra-block error variance.

The estimates of the varietal differences derived from the inter-block comparisons are similarly given by the differences of

$$rT_s/(r-\lambda) \quad \text{or} \quad rT'_s/(r-\lambda),$$

in units of the total yield of $r$ replicates. The error variance of these sets of quantities is $kr^2/(r-\lambda)w'$, where $1/w'$ is the error variance of the inter-block comparisons, in units of a single plot.

The most efficient estimates of the varietal differences are given by the quantities

$$Y_s = V_s + \mu W_s$$

where

$$W_s = (v-k)V_s - (v-1)T_s + (k-1)G,$$

and

$$\mu = \frac{w-w'}{wv(k-1)+w'(v-k)}.$$

The error variance of the $Y$'s is

$$\frac{kr(v-1)}{wv(k-1)+w'(v-k)}.$$

The analysis of variance follows the lines shown in the accompanying table, in which $dev^2$ indicates the sum of squares of the deviations, $y$ the individual yields and $B$ the block totals. If recovery of inter-block information is not required then only the part (*a*) of the analysis of variance is required. There is also no need to subdivide the sum of squares for blocks, ignoring varieties. The comparison of the mean square (*a*) for varieties with intra-block error gives an exact test of significance for the intra-block estimates.

| Method (*a*). | D.F. | S.S. (*a*). | | S.S. (*b*). | Method (*b*). |
|---|---|---|---|---|---|
| Blocks (ignoring varieties) : | | | | | Blocks (eliminating varieties) : |
| Varietal component . . | $v-1$ | $\frac{dev^2 T}{k(r-\lambda)}$ | | $\frac{dev^2 W}{rv(v-k)(k-1)}$ | Varietal component |
| Remainder . . . . | $b-v$ | † | → | † | Remainder |
| Total . . . . . | $b-1$ | $\frac{dev^2 B^*}{k}$ | | † | Total |
| Varieties (eliminating blocks) . | $v-1$ | $\frac{dev^2 kQ}{k^2 rE}$ | | $\frac{dev^2 V}{r}$ | Varieties (ignoring blocks) |
| Intra-block error . . . | $rv-v-b+1$ | † | ←→ | † | Intra-block error |
| Total . . . . . | $rv-1$ | $dev^2 y^*$ | ←→ | $dev^2 y^*$ | Total |

* Requires checking. † Calculated by addition or subtraction.

If the mean square for intra-block error is $M$ and the mean square (*b*) for the $b-1$ degrees of freedom for blocks, eliminating varieties, is $M''$, $w$ and $w'$ are estimated from the equations

$$w = \frac{1}{M}, \quad w' = \frac{v(r-1)}{k(b-1)M'' - (v-k)M}.$$

If $M''$ is less than $M$, $w'$ may ordinarily be taken as equal to $w$.

If the structure of the design is such that the blocks fall into $r$ groups, each containing a single replication of all the varieties (as in solutions Nos. 6, 7, 21 and 25), and each of these groups of blocks is itself arranged in a large block, then the sum of squares corresponding to the $r-1$ degrees of freedom for complete replications must be eliminated from the sums of squares for blocks. The formula for $w'$ will then become

$$w' = \frac{r-1}{rM''-M}.$$

In the more general case in which there are $c$ such groups of blocks, each containing $r/c$ replications, we have

$$w' = \frac{v(r-1)-k(c-1)}{k(b-c)M''-(v-k)M}.$$

In the case of a Youden square the sum of squares corresponding to the $k-1$ degrees of freedom for the $k$ series (representing leaf position, etc.) must be eliminated from the intra-block error.

*Example* 8

Table 3 shows the scores of 18 litters of 4 rats in a discrimination test, the values shown being the square roots of the numbers of trials prior to 12 consecutive successful trials (the data are a random selection from scores published by F. A. E. Crew (10)). A dummy trial of 9 treatments (*e.g.* dietary treatments) in blocks of 4, corresponding to litters, is superimposed on these scores. (Design 11 of Table XVII, $r = 8$, $v = 9$, $k = 4$, $b = 18$, $\lambda = 3$.)

TABLE 3. SCORES OF 18 LITTERS OF RATS

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| *f* 2·6 | *f* 5·9 | *a* 7·0 | *i* 2·4 | *i* 5·0 | *d* 10·1 | *b* 3·9 | *h* 4·0 | *b* 2·8 |
| *d* 9·7 | *g* 2·6 | *f* 4·6 | *d* 4·0 | *h* 7·4 | *a* 9·7 | *d* 4·1 | *f* 6·1 | *f* 2·6 |
| *c* 5·4 | *i* 5·9 | *i* 4·9 | *g* 3·0 | *e* 10·3 | *f* 5·7 | *e* 6·4 | *g* 4·4 | *e* 2·8 |
| *e* 6·9 | *b* 6·3 | *c* 3·3 | *f* 2·4 | *c* 9·4 | *h* 7·5 | *i* 6·3 | *c* 3·3 | *h* 3·3 |
| 24·6 | 20·7 | 19·8 | 11·8 | 32·1 | 33·0 | 20·7 | 17·8 | 11·5 |
| *b* 5·7 | *b* 4·7 | *a* 3·0 | *c* 7·5 | *c* 3·7 | *i* 3·0 | *d* 4·5 | *g* 2·6 | *b* 7·3 |
| *h* 9·3 | *g* 6·6 | *h* 1·4 | *g* 2·2 | *a* 5·2 | *g* 2·6 | *b* 6·0 | *e* 4·9 | *e* 5·4 |
| *c* 5·4 | *a* 5·5 | *i* 4·2 | *e* 2·6 | *d* 2·4 | *a* 4·7 | *g* 4·6 | *d* 6·0 | *f* 5·7 |
| *i* 6·1 | *h* 5·3 | *d* 2·8 | *a* 4·4 | *b* 2·4 | *e* 2·4 | *c* 3·3 | *h* 4·6 | *a* 4·4 |
| 26·5 | 22·1 | 11·4 | 16·7 | 13·7 | 12·7 | 18·4 | 18·1 | 22·8 |

Grand Total : 354·4. Mean : 4·9$\dot{2}$.

Table 4 shows the values of $V$, $T$, $4Q$ and $W$ for the nine treatments *a*-*i*. The analysis of variance is shown in Table 5.

TABLE 4. CALCULATION OF ADJUSTED SCORES IN DISCRIMINATION TEST

| | V. | T. | 4Q = 4V−T. | Q/E+$\frac{1}{9}$G. | W = 5V−8T+3G. | Y = V+$\mu$W. |
|---|---|---|---|---|---|---|
| *a* | 43·9 | 152·2 | +23·4 | 46·3 | +65·1 | 45·7 |
| *b* | 39·1 | 156·4 | 0 | 39·4 | +7·5 | 39·3 |
| *c* | 41·3 | 169·6 | −4·4 | 38·1 | −87·1 | 38·9 |
| *d* | 43·6 | 151·7 | +22·7 | 46·1 | +67·6 | 45·4 |
| *e* | 41·7 | 159·2 | +7·6 | 41·6 | −1·9 | 41·6 |
| *f* | 35·6 | 162·0 | −19·6 | 33·6 | −54·8 | 34·1 |
| *g* | 28·6 | 138·3 | −23·9 | 32·3 | +99·8 | 31·3 |
| *h* | 42·8 | 172·5 | −1·3 | 39·0 | −102·8 | 40·0 |
| *i* | 37·8 | 155·7 | −4·5 | 38·0 | +6·6 | 38·0 |
| | 354·4 | 1417·6 | 0 | 354·4 | 0 | 354·3 |
| Divisor | 8 | 4·5<br>= 20 | $4^2{\cdot}8{\cdot}27/32$<br>= 108 | | 8·9·5·3<br>= 1080 | |

TABLE 5. ANALYSIS OF VARIANCE, DISCRIMINATION TEST

| | D.F. | S.S. (*a*). | S.S. (*b*). | M.S. |
|---|---|---|---|---|
| Blocks : | | | | |
| Varietal component . | 8 | 41·4684 | 37·0634 | 4·6329 (*b*) |
| Remainder . . | 9 | 138·2011 | 138·2011 | 15·3557 |
| Total . . . | 17 | 179·6695 | 175·2645 | 10·3097 (*b*) |
| Varieties . . . | 8 | 19·6044 | 24·0094 | 2·4506 (*a*) |
| Error . . . . | 46 | 119·4506 | 119·4506 | 2·5968 |
| Total . . . . | 71 | 318·7245 | 318·7245 | — |

From the results of the analysis of variance we obtain

$$w = \frac{1}{2{\cdot}5968} = 0{\cdot}3851, \quad w' = \frac{63}{68\times 10{\cdot}3097 - 5\times 2{\cdot}5968} = 0{\cdot}0916,$$

$$\mu = \frac{0{\cdot}3851 - 0{\cdot}0916}{27\times 0{\cdot}3851 + 5\times 0{\cdot}0916} = \frac{0{\cdot}2935}{10{\cdot}8557} = 0{\cdot}02704.$$

The final adjusted scores in terms of the total scores of eight rats are given in the last column of Table 4. The standard error of these scores is

$$\sqrt{\frac{256}{10{\cdot}8557}} = \sqrt{23{\cdot}58} = 4{\cdot}86.$$

The scores based on intra-block information only (which are equal to $Q/E+\frac{1}{9}G$) are also shown in Table 4. Their standard errors are

$$\sqrt{(8\times 2{\cdot}5968\times 32/27)} = \sqrt{24{\cdot}62} = 4{\cdot}96.$$

Thus the gain in information from the recovery of the inter-block information is $24{\cdot}62/23{\cdot}58 - 1$ or 4·4 per cent. (excluding losses due to inaccuracy of weighting). The gain is here trivial, but if inter-litter and intra-litter comparisons had been of equal accuracy, the gain would have been 18·5 per cent.

The litter totals can be adjusted to allow for the effects of the treatments by deducting one-eighth the scores based on intra-litter differences only of each of the four treatments contained in the litter concerned from the total of that litter (adding four times the general mean if desired). Thus the adjusted total of the first litter is $24{\cdot}6 - 4{\cdot}20 - 5{\cdot}76 - 4{\cdot}76 - 5{\cdot}20 + 4\times 4{\cdot}92 = 24{\cdot}36$.

## TABLES XX AND XXI. ORDINAL (OR RANKED) DATA

It is often necessary to draw statistical conclusions from data giving the order of a number of magnitudes without knowledge of their quantitative values. Thus in tests of psychological preference subjects can often express preferences, without being able to assign numerical values to the force with which the preference is felt. Not infrequently, also, an experimenter who possesses quantitative values may suspect that the metric used is unsuitable to the comparisons he wishes to make, and prefer to draw conclusions only from the order of the magnitudes observed.

The analysis of such data is greatly facilitated by Table XX, which gives the average deviate of the $r$th largest of samples of $n$ observations drawn from a normal distribution having unit variance. Symbolically this comes to

$$\mu_r = \int_{-\infty}^{\infty} \frac{n!}{(r-1)!(n-r)!}\, p^{n-r} q^{r-1} xz\, dx,$$

where $z$ is the ordinate of the normal curve, and $p$ and $q$ are the probabilities respectively of falling short of and exceeding $x$.

From the order in which any subject places a series of objects scores may now be assigned. With a number of different subjects, the experimenter is then in a position to test, for example, whether the scores assigned to any one object significantly exceed those assigned to another; or, more generally, to apply an analysis of variance to determine whether the variance in score among objects exceeds the remainder due to differences in the order chosen by different subjects. More elaborate analyses afford tests of differentiation in preference between classes of subjects of different sex or age. The labour of summing the squares of individual scores is saved by using the sums of squares given in Table XXI.

The scoring of ties causes little difficulty. If, for example, the fifth and sixth objects out of 13 are judged to be equal, instead of scoring one ·39 and the other ·19, each is given the same score ·29. In such cases the sum of squares needs adjustment. When only two objects are judged to be equal it is easy to see that the reduction is $\frac{1}{2}(\cdot 39-\cdot 19)^2 = \cdot 02$, so that the sum of squares is 10·7904 in place of 10·8104 shown by the table.

The table can also be used to obtain the factor by which the range of a sample of size $n$ should be multiplied to give an estimate of the standard deviation of an observation. The factor is in fact $1/2\mu_1$. Thus the range of a sample of 8 should be multiplied by $1/(2\times 1\cdot 42) = 0\cdot 352$. Similarly if the mean of the difference between the largest and smallest observations, and between the second largest and second smallest is taken, the appropriate factor is $1/\{2\times\frac{1}{2}(\mu_1+\mu_2)\}$.

In a normal distribution the range provides quite an efficient estimate of the standard deviation if the sample size is small, but the efficiency becomes low as the size of the sample is increased, as is shown by the following values:

| Size. | Efficiency. | Size. | Efficiency. |
|---|---|---|---|
| 2 | 100 | 15 | 76·8 |
| 4 | 97·5 | 20 | 69·9 |
| 6 | 93·3 | 25 | 64·7 |
| 8 | 89·0 | 50 | 48·9 |
| 10 | 85·0 | 100 | 34·8 |

Moreover if the distribution is not normal the use of the factors appropriate to the normal distribution will give biased estimates of the standard deviation, whereas the ordinary estimate of the variance based on the sum of the squares of the deviations will be unbiased whatever the form of the distribution. The magnitude of the bias tends to increase as the size of the sample increases.

On both grounds, therefore, the range should not be used with large samples. Nor has its use with a single small sample much to recommend it, since in this case the calculation of the ordinary estimate is not laborious. Similar objections apply to the use of a modified $t$ test based on the range. If a large number of small samples of the same size (say not greater than 10) from a distribution are available, however, the mean range will provide a reasonably efficient estimate of the standard deviation and one that is not likely to be seriously biased unless the distribution departs markedly from normality. The range can also be appropriately used in quality control work to provide a control on the variability, though in this case there is no point in converting to the standard deviation, provided the sample size is constant.

## TABLE XXII. THE LEADING DIFFERENCES OF POWERS OF NATURAL NUMBERS

The table gives values of $\Delta^r 0^s/r!$, where $\Delta^r 0^s$ denotes the leading $r$th difference of the $s$th powers of the natural numbers commencing at zero. Its application to combinatorial problems arises from the identity

$$\frac{\Delta^r 0^s}{r!} \equiv \sum_{s/r} \frac{s!}{(p_1!)^{\pi_1}\pi_1!(p_2!)^{\pi_2}\pi_2!\ \ldots}$$

where $(p_1^{\pi_1}\ p_2^{\pi_2}\ \ldots)$ is a partition of the number $s$ into $r$ parts, *i.e.*

$$\Sigma\pi p = s, \qquad \Sigma\pi = r.$$

The partitional function on the right represents the number of ways of dividing $s$ persons into the $r$ parties specified by the partition. The summation proceeds over all partitions of $s$ into $r$ parts.

Hence $\Delta^s 0^s/r\,!$ is the number of ways of dividing $s$ persons into $r$ parties of all possible sizes. For example, with $s = 4$, $r = 2$, the tables gives 7 ways, which are in fact,

$$(ab)(cd),\ (ac)(bd),\ (ad)(bc),\ (abc)(d),\ (abd)(c),\ (acd)(b),\ (bcd)(a).$$

If $s$ objects are equally likely to fall into any of $n$ classes, the probability that they will fall into exactly $r$ of the possible $n$ classes can be shown to be

$$\frac{1}{n^s}\,\frac{n!}{(n-r)!}\,\frac{\Delta^r 0^s}{r!}.$$

Hence the distribution of $r$ can be derived from the table. Thus the table is useful for testing whether objects fall together in classes too often or too rarely. For practical applications see Stevens (12), (13).

### *Example* 9

In the expression for $\pi$ it may be noticed that in the first 20 decimal places all digits save zero occur at least once. Zero is the only digit absent. It is not obvious how frequently 0, 1, 2,. . . absent digits would be found in a series chosen at random.

Putting $s=20$, $n=10$, $r=9$ in the expression shown we find

$$\frac{10!}{10^{20}}\times 1201\cdot 13\times 10^{10}$$
$$=3\cdot 6288\times\cdot 120113=\cdot 4359.$$

Hence in 43·59 per cent. of trials we should expect exactly one absent digit; similarly we should expect no digit absent in 21·47 per cent. of such trials; consequently two or more digits will be absent in the remaining 34·94 per cent. The absence of one digit out of ten is the commonest event in such trials.

---

The table may be extended by using the recurrence relation

$$\frac{\Delta^{r+1}0^{s+1}}{(r+1)!}=(r+1)\frac{\Delta^{r+1}0^{s}}{(r+1)!}+\frac{\Delta^{r}0^{s}}{r!}.$$

This equation may be written

$$\frac{\Delta^{r+1}0^{s+1}}{(r+1)!}=\frac{\Delta^{r}(\Delta+1)0^{s}}{r!}=\frac{\Delta^{r}1^{s}}{r!}$$

showing that the entries derived from powers of natural numbers commencing at *unity*, may be read from this same table, with $r$ and $s$ both increased by one.

## TABLE XXIII. ORTHOGONAL POLYNOMIALS

If a polynomial regression line

$$Y=a+bx+cx^2+dx^3+\ldots$$

is to be fitted to a series of $n'$ observations at equal intervals of $x$, we may conveniently fit instead the equivalent line

$$Y=A+B\xi_1+C\xi_2+D\xi_3+\ldots$$

where $\xi_1$, $\xi_2$, $\xi_3$, . . . are orthogonal polynomials of degree 1, 2, 3 . . . ($A=\bar{y}$ and $\xi_1=x-\bar{x}$). A method of determining the coefficients by successive addition, and also of calculating the polynomial values by the same process, is given in *Statistical Methods*. If the numerical values of the $\xi$'s (or multiples $\xi'=\lambda\xi$ of them) are known for each value of $x$ an alternative procedure (set out below) is available. Table XXIII shows these numerical values for all $\xi'$ from $\xi'_1$ to $\xi'_5$, and all $n'$ from 3 to 75. In each case the smallest value of $\lambda$ is chosen which gives integral values to $\xi'$. This value is shown at the foot of the corresponding column, and is the coefficient of the highest power of $x$ in $\xi'$. The sum of squares of each set of $\xi'$'s is also shown. Above $n'=8$ only the values for positive $x-\bar{x}$ are given. More extensive tables are available elsewhere (38, 39). The first reference (38) gives values of $\xi'_1$ to $\xi'_5$ for all $n'$ to 104, the second (39) gives values of $\xi'_1$ to $\xi'_9$ for all $n'$ to 52.

The coefficient of each $\xi'$ in the regression equation is found by calculating the sum of the products of the observed values $y$ and the corresponding values of $\xi'$, and dividing by the sum of squares of $\xi'$. Thus the coefficient of $\xi'_2$ is

$$C'=\frac{S(y\xi'_2)}{S(\xi'^2_2)}.$$

The corresponding sum of squares in the analysis of variance is

$$C'.S(y\xi'_2)=\frac{\{S(y\xi'_2)\}^2}{S(\xi'^2_2)}$$

and the estimated standard error of $C'$ is $s/\sqrt{S(\xi'^2_2)}$.

If any or all of the values of $Y$ corresponding to the $n'$ observations are required they may be calculated directly from the values of the coefficients and the $\xi'$'s, using the equation

$$Y=A+B'\,\xi'_1+C'\,\xi'_2+D'\,\xi'_3+\ldots$$

This procedure is particularly useful when only a few values are required.

If the regression equation is required in terms of powers of $x$ the formulæ for the $\xi$'s in terms of powers of $x$ given in *Statistical Methods* (editions 3-6), or the recurrence formula

$$\xi_{r+1} = \xi_1\xi_r - \frac{r^2(n'^2-r^2)}{4(4r^2-1)}\,\xi_{r-1}$$

($\xi_0 = 1$) may be used, together with $\xi' = \lambda\xi$. Alternatively the coefficients of the powers of $x$ in each $\xi'$ may be deduced from the tabulated values of the $\xi'$'s, as illustrated in the example below (remembering that the even $\xi'$'s contain only even powers of $x$, and the odd $\xi'$'s odd powers of $x$).

The relative advantages of the above method and the alternative process of successive addition depend on the computing facilities available, but with a good multiplying machine, and no printing adding machine, the use of the $\xi'$'s will be found to be decidedly more expeditious, especially when $n'$ is small. If polynomials of the fourth or fifth degree are to be fitted the sums and differences of the pairs of observations (commencing from the centre) should be tabulated, thereby halving the number of multiplications. At the same time the values may frequently be reduced in magnitude by the deduction of a constant amount, and if necessary by rounding off. If polynomials have to be fitted to a number of sets of observations with the same $n'$ it is worth tabulating $\xi'_1+\xi'_3+\xi'_5$ and $\xi'_2+\xi'_4$, and using the checks $Sy(\xi'_1+\xi'_3+\xi'_5) = Sy\xi'_1+Sy\xi'_3+Sy\xi'_5$ and $Sy(\xi'_2+\xi'_4) = Sy\xi'_2+Sy\xi'_4$.

### *Example* 10

Table 6 shows the yields (bushels per acre) of plots 9 and 7 of Broadbalk wheat field for the 30 years 1855-1884. The only difference in manurial treatment was that plot 9 received nitrate of soda and 7 an equivalent quantity of nitrogen as sulphate of ammonia. Investigate the slow changes in the difference between the two plots by fitting a polynomial of the fifth degree. (This example is discussed in *Statistical Methods*, Sections 26 and 28.1. The values there given have been rounded off to one decimal place.)

TABLE 6

| Harvest Year. | 9. | 7. | 9-7. | Harvest Year. | 9. | 7. | 9-7. | Harvest Year. | 9. | 7. | 9-7. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1855 | 29·6 | 33·0 | −3·4 | 1865 | 44·1 | 40·2 | +3·9 | 1875 | 30·5 | 26·6 | +3·9 |
| 1856 | 32·4 | 36·9 | −4·5 | 1866 | 32·5 | 29·9 | +2·6 | 1876 | 33·3 | 25·5 | +7·8 |
| 1857 | 43·8 | 44·8 | −1·0 | 1867 | 29·1 | 22·2 | +6·9 | 1877 | 40·1 | 19·1 | +21·0 |
| 1858 | 37·6 | 38·9 | −1·3 | 1868 | 47·8 | 39·2 | +8·6 | 1878 | 37·2 | 32·2 | +5·0 |
| 1859 | 30·0 | 34·7 | −4·7 | 1869 | 39·0 | 28·2 | +10·8 | 1879 | 21·9 | 17·2 | +4·7 |
| 1860 | 32·6 | 27·7 | +4·9 | 1870 | 45·5 | 41·4 | +4·1 | 1880 | 34·1 | 34·3 | −·2 |
| 1861 | 33·8 | 34·9 | −1·1 | 1871 | 34·4 | 22·3 | +12·1 | 1881 | 35·4 | 26·1 | +9·3 |
| 1862 | 43·4 | 35·9 | +7·5 | 1872 | 40·7 | 29·1 | +11·6 | 1882 | 31·8 | 34·8 | −3·0 |
| 1863 | 55·6 | 53·7 | +1·9 | 1873 | 35·8 | 22·8 | +13·0 | 1883 | 43·4 | 36·3 | +7·1 |
| 1864 | 51·1 | 45·8 | +5·3 | 1874 | 38·2 | 39·6 | −1·4 | 1884 | 40·4 | 37·8 | +2·6 |

The computations are given in Table 7. The first step is to form the sums and differences of pairs of values, working outwards from the centre pair.

Thus $$+14{\cdot}9 = +4{\cdot}1+10{\cdot}8 \quad\text{and}\quad -6{\cdot}7 = +4{\cdot}1-10{\cdot}8.$$

The totals of the sums and the differences check against the sum and difference of the totals of the first fifteen and last fifteen of the original values. The total sum of squares of the deviations from the mean, 1016·99, can be calculated from the sums and differences, and, as a check, from the original values, at the same time as these totals are obtained.

The sums of products with the $\xi'$ values are next calculated, and at the same time the regression coefficients and the sum of squares accounted for by each term. The sums of products may be checked by direct recalculation or by the method described above. Finally the analysis of variance can be completed, giving the residual mean square of 24·14. It will be seen the first two terms account for a substantial part of the variation, but the mean squares of the remaining three terms are all below the residual mean square. Thus a parabola adequately describes the slow changes.

The equation of the parabola is

$$Y = 4{\cdot}4667+0{\cdot}13255\,\xi'_1-0{\cdot}029762\,\xi'_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

TABLE 7

| Years. | Sum. | Difference. |
|---|---|---|
| '70, '69 | +14·9 | −6·7 |
| '71, '68 | +20·7 | +3·5 |
| '72, '67 | +18·5 | +4·7 |
| '73, '66 | +15·6 | +10·4 |
| '74, '65 | +2·5 | −5·3 |
| '75, '64 | +9·2 | −1·4 |
| '76, '63 | +9·7 | +5·9 |
| '77, '62 | +28·5 | +13·5 |
| '78, '61 | +3·9 | +6·1 |
| '79, '60 | +9·6 | −0·2 |
| '80, '59 | −4·9 | +4·5 |
| '81, '58 | +8·0 | +10·6 |
| '82, '57 | −4·0 | −2·0 |
| '83, '56 | +2·6 | +11·6 |
| '84, '55 | −0·8 | +6·0 |
| | +134·0<br>4·46 | +61·2 |

Sums of Products.

| Term. | $S\xi'y$. | Regression Coefficient. | Sum of Squares. |
|---|---|---|---|
| 1 | +1191·6 | +0·1325473 | 157·9434 |
| 2 | −8990·0 | −0·02976190 | 267·5595 |
| 3 | −8772·2 | $-0{\cdot}0^{3}4106789$ | 3·6026 |
| 4 | +148509·0 | $+0{\cdot}0^{4}4044817$ | 6·0069 |
| 5 | +72400·4 | $+0{\cdot}0^{4}3374157$ | 2·4429 |

Analysis of Variance.

| | Degrees of Freedom. | Sum of Squares. | Mean Square. |
|---|---|---|---|
| Polynomial terms . | 5 | 437·55 | 87·51 |
| Remainder . . | 24 | 579·44 | 24·14 |
| Total . . | 29 | 1016·99 | |

which can be converted into an equation in $x$, if this is required, by substitution for $\xi'_1$ and $\xi'_2$. If $x$ is measured from the centre point of the series $\xi'_1 = 2x$ and $\xi'_2 = \frac{3}{2}x^2 + C$, the coefficients 2 and $\frac{3}{2}$ of the highest powers of $x$ being given in Table XXIII. Further, since $\xi'_2 = -112$ when $x = +\frac{1}{2}$, we have immediately $C = -112{\cdot}375$.

Alternatively the recurrence formula gives

$$\xi'_2 = \frac{3}{2}\xi_2 = \frac{3}{2}\left[\xi_1 \,.\, \xi_1 - \frac{1^2 \,.\, 899}{4{\cdot}3} \,.\, 1\right] = \frac{3}{2}x^2 - 112{\cdot}375.$$

This gives the equation

$$Y = 7{\cdot}8112 + 0{\cdot}26510x - 0{\cdot}044643x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The value of $Y$ for any given year can be calculated from (1). Thus for the year 1855

$$Y = 4{\cdot}4667 + 0{\cdot}13255\,(-29) - 0{\cdot}029762\,(+203) = -5{\cdot}4189.$$

Similarly for the year 1884 the value is +2·2690. Equation (2) will, of course, give the same values.

The parabola has a maximum at $x = +2{\cdot}969$, *i.e.* at the date 1872·469, $Y$ then having the value +8·2048. The improvement of plot 9 relative to plot 7 therefore appears to have reached a climax at about this date, there being some evidence of a reduction in the difference subsequently.

As a further exercise polynomials may be fitted to the yields of the two plots separately. The difference of these two polynomials clearly gives the polynomial already determined.

## TABLE XXIV. CALCULATION OF INTEGRALS FROM EQUALLY SPACED ORDINATES

The table supplies data useful for the mechanical integration of functions calculated at not very closely spaced ordinates. Its use may be illustrated by the series of reciprocals of numbers from 15 to 19. In this case the integral is known to be $\log_e 19/15$, or ·23638,87780,66; the average ordinate over the four units in the panel is, therefore, ·05909,71945,16. An accurate calculation can be obtained from the series of reciprocals and their even differences. Taking enough to use the eighth difference, these are as follows to ten places of decimals :—

| Number. | Reciprocal. | Differences. | | | |
|---|---|---|---|---|---|
| | | 2nd. | 4th. | 6th. | 8th. |
| 13 | ·07692,30769 | | | | |
| 14 | ·07142,85714 | 73,26008 | | | |
| 15 | ·06666,66667 | 59,52380 | 3,23209 | | |
| 16 | ·06250,00000 | 49,01961 | 2,33426 | 28353 | |
| 17 | ·05882,35294 | 40,84968 | 1,71996 | 18438 | 3745 |
| 18 | ·05555,55556 | 34,39971 | 1,29004 | 12268 | |
| 19 | ·05263,15789 | 29,23978 | 98280 | | |
| 20 | ·05000,00000 | 25,06265 | | | |
| 21 | ·04761,90476 | | | | |

Using the coefficients for 4 sub-panels in the table for central ordinate and even differences, the successive approximations obtained are as follows :—

| | Correction. | Value. | Error. |
|---|---|---|---|
| No difference | | ·05882,35294,00 | −27,36651,16 |
| 2nd ,, | +27,23312,00 | ·05909,58606,00 | −13339,16 |
| 4th ,, | +13377,47 | ·05909,71983,47 | +38,31 |
| 6th ,, | −39,02 | ·05909,71944,45 | −71 |
| 8th ,, | +86 | ·05909,71945,31 | +15 |

To show the adequacy of the adjustments two additional figures have been retained. The evaluation is satisfactory using only 6th differences, although the 8th difference is a four-figure number.

The formulæ using integral multipliers in the first part of the table (4 sub-panels) give exactly equivalent values to the use of 4th and 6th differences.

## TABLES XXV TO XXXII AND XXXIV. LOGARITHMS, TRIGONOMETRICAL FUNCTIONS, ETC.

These tables comprise the standard functions that are likely to be required by the statistician. Five-figure accuracy has been adopted in most of the tables, since four-figure tables are already sufficiently plentiful, and five figures bridge the gap between the slide-rule (roughly three-figure accuracy) and seven-figure standard tables.

In addition to the ordinary table of logarithms to base 10 of numbers 1·00 to 9·99 (Table XXV) six pages of natural logarithms ($\log_e$ ·100—$\log_e$ 99·9) are included (Table XXVI). This table will prove useful to those who find it convenient to work with natural logarithms. Exact squares of all three-figure numbers (Table XXVII) will enable an occasional analysis of variance to be performed without a machine. Square roots (Table XXVIII) are so tabulated that the square roots of each three-figure number and of ten times that number appear together. This facilitates reference to the table, and serves as a reminder of the necessity of considering the position of the decimal point before taking out a square root. Reciprocals (Table XXIX) are given to six figures, since interpolation is not likely to be required in this table when a machine is available, and the additional figure is useful when division is being performed on a machine by means of reciprocals.

Table XXX gives the factorials of the numbers 1-300 to six significant figures, and their logarithms to seven decimal places. Of the trigonometrical functions only natural sines (Table XXXI) and tangents (Table XXXII) have been included. By tabulating $(90° - \theta) \tan \theta$ instead of $\tan \theta$ for values of $\theta$ greater than 60° five-figure accuracy of interpolated values is assured, even for high angles.

In Table XXXIV will be found a collection of constants, conversion factors, etc. Only the English, American and metric systems of weights and measures have been included, the more uncommon measures of these systems being given in their most easily remembered forms.

The tabulation of the above tables to 5 and 6 figure accuracy does not preclude their use to four-figure accuracy only when the circumstances warrant it. For this purpose they will be found little slower than the ordinary four-figure tables, since the amount of page-turning is the same. When using logarithms to four-figure accuracy it will frequently be found most expeditious, as well as slightly more accurate, to retain 5 figures in the logarithmic calculations, without making any attempt to interpolate to more than four-figure accuracy. General instructions for interpolation are given in the last section of the Introduction.

## TABLE XXXIII. RANDOM NUMBERS

This table contains 7500 two-figure random numbers arranged on six pages. The method of construction of the table, and the tests made on it, are described in the first edition. Methods of using the table may best be made clear by a couple of examples.

*Example* 11. Required to select a series of numbers at random from the numbers 1-16.

The simplest way of doing this is to select a row, column or diagonal of two-figure numbers (such selection being made without previous inspection of the numbers themselves), and pick out all numbers between 1 and 16 as they occur. With a number like 16, however, such a procedure is tedious and wasteful of numbers. As an alternative we may divide each two-figure number by 16 and take the remainder (0 counting as 16). It is clear that to give all numbers an equal chance of selection the numbers 97, 98, 99 and 00 must be rejected. A modification of this process, which somewhat lightens the arithmetic, is to use the divisor 20 instead of 16, rejecting any number which gives 00, 17, 18 or 19. Thus 73 would be read as 13.

*Example* 12. Required to arrange 16 treatments, numbered 1-16, in random order.

We may start by selecting a series of numbers between 1 and 16 by one of the methods of the last example, rejecting any number if it occurs a second time. After half the numbers have been selected, however, the process becomes tedious, since the rejections become more numerous, on the average, than the selections. This difficulty may be overcome by choosing the first number by making a selection between 1 and 16, the second by making a selection between 1 and 15, the third by making a selection between 1 and 14, and so on, the actual number selected at each stage being taken to indicate (by counting from one end of the series) which of the numbers still remaining is to be chosen. This operation can easily be performed by writing down all the numbers (or other symbols) in order and striking out each as it is selected. Thus, taking the fourteenth column of two-figure numbers on page IV, which reads

82, 05, 30, 58, 55, 66, 53, 16, 89, 56, 16, 70, 70, 07, 37,

we obtain on dividing successively by 16, 15, 14, 13 . . . the following series of remainders

2, 5, 2, 6, 7, 11, 3, 7, 1, 7, 4, 5, 2, 1, 1.

We therefore select the second, fifth, second, etc., numbers of all numbers still unselected. This gives

2, 6, 3, 9, 11, 16, 5, 13, 1, 15, 10, 14, 7, 4, 8, 12.

Various combinations of the above methods can also be used. An excellent way of proceeding in Example 12 is to select the first 7 treatments by selecting a number between 1 and 16 by dividing by 20, as suggested in Example 11, rejecting any number which occurs a second time, afterwards completing the selection by division of the subsequent two-figure numbers by decreasing divisors according to the method just given. Using this process on the numbers printed above we obtain as the first 7 treatments

2, 5, 10, 18 (rejected), 15, 6, 13, 16.

Division of the subsequent numbers by 9, 8, 7, etc., gives the series 8, 8, 2, etc., so that the corresponding treatments are the eighth, eighth, second, etc., of those remaining, *i.e.* 12, 14, 3, etc.

## TABLES XXXIII1 AND XXXIII2

Table XXXIII1 gives 750 random permutations, numbered 0-749, of the numbers 0-9. By its aid random permutations of up to 10 objects can be obtained immediately, instead of by the procedure given in Example 12. Table XXXIII2 gives 200 similar permutations of the numbers 0-19. It can also be used to give additional permutations of the numbers 0-9 which are independent of one another and of those given in Table XXXIII1, by taking separately the numbers 0-9 and 10-19 of each permutation. These tables are useful in such processes as the allocation of treatments to the plots of a randomized block experiment, the randomization of the rows, columns and letters of a Latin square, etc.

Each permutation of 10 is arranged in two groups of 5 printed one beneath the other, and each permutation of 20 is similarly arranged in a 4×5 pattern. The reading of the permutation in a different order, *e.g.* down instead of across, will give another random permutation, but not one that is fully independent of the first.

For this reason, when large numbers of random and strictly independent permutations are required they should be generated from a table of random numbers, or by some other fully random process. For the purpose of experimental design, however, Tables XXXIII1 and XXXIII2 will provide all that is normally required.

Numbering of the permutations serves two purposes. It enables permutations to be selected from the table at random by selecting a number at random in the appropriate range from Table XXXIII. It also enables the permutations which are to be used in particular experiments to be specified, thus ensuring that the arrangement used is really random, without drawing up the detailed plans.

The tables can also be used to give random permutations of more than 20 numbers by one of the methods illustrated in Examples 12.1 and 12.2. The first method is applicable to other symbols, such as letters, without any additional transcription.

The tables were generated by Mr B. M. Church from Table XXXIII by the aid of Hollerith equipment. Tables of 1000 permutations of 9 and 1000 of 16 have previously been published by Cochran and Cox (3b), following a suggestion by G. W. Snedecor.

*Example* 12.1. Arrange the 40 symbols *A-Z*, *a-n*, in random order.

By means of Table XXXIII the symbols are first divided into two groups in such a manner that each symbol has an equal chance of falling in either group. If at the end of the process one group contains more than 20 symbols a random selection is made for transfer to the other group, so that two groups of 20 are obtained. These groups are then permuted by means of two permutations from Table XXXIII2.

Thus using column no. 38 of Table XXXIII (IV), and placing odds in group I, we obtain

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Group I . . | *A* | (*B*) | *D* | *E* | *F* | *G* | (*H*) | *I* | *J* | *K* | *L* | *N* | *P* | *S* | *X* | *Z* | *a* | *b* | *g* | *i* | *j* | *m* |
| Group II . . | *C* | *M* | *O* | *Q* | *R* | *T* | *U* | *V* | *W* | *Y* | *c* | *d* | *e* | *f* | *h* | *k* | *l* | *n* | (*H*) | (*B*) | | |

Selection of two numbers at random from 1-22 gives nos. 7 and 2, *i.e.* symbols *H* and *B*, for transfer. The blanks thereby created in group I are filled by the symbols *j* and *m*. Transferred symbols are shown bracketed. Permutations number 191 and 100 applied to the two groups, 0 being read as 20, then give the final permutation :—

*a X J g E j b N D S A I P m F i K G Z L l U O e d k h C Y f M H V Q R B W c T n*

It should be noted that transfers are only necessary to reduce the group size to within range of the tables. Thus in arranging 32 symbols in random order by subdivision into two groups it will only rarely happen that either group exceeds 20.

For numbers of symbols larger than 40 three or more groups may be used. If transfer is necessary it is immaterial in which group a transferred symbol is placed, provided the permissible size is not exceeded.

*Example* 12.2. Arrange the numbers 1-24 in random order.

Instead of dividing into groups we may, when the total number of symbols does not greatly exceed 20, insert the extra symbols at randomly located points in a random permutation of the first 20 symbols. This is easily done if the symbols are numbers. Permutation no. 54 gives

2, 11, 20, 1, 7, 19, 13, 18, 8, 6, 9, 15, 3, 17, 16, 4, 12, 5, 10, 14.

A number, say 17, is then selected at random from the numbers 1-21, and 21 is therefore inserted immediately before number 17. If 21 were selected the number 21 would be inserted at the end of the series. The process is repeated by selecting at random from the numbers 1-22, 1-23 and 1-24 in turn. If the numbers 7, 23, 12 are obtained the final arrangement is

2, 11, 20, 1, 22, 7, 19, 13, 18, 8, 6, 9, 15, 3, 21, 17, 16, 4, 24, 12, 5, 10, 14, 23.

## INTERPOLATION

Many of the tables are provided with special aids to rapid interpolation, such as proportional parts, mean differences, and tabular differences printed in the tables. In other cases linear interpolation is generally sufficient. Thus if the table gives values for $u_x$ and $u_{x+1}$, and the value is required for $u_{x+\theta}$, we may take

$$u' = \theta u_{x+1} + \phi u_x,$$

where $\phi$ stands for $1-\theta$. This form is convenient for machine work; mentally it is often quicker to use the differences between the tabular values and calculate

$$u' = u_x + \theta(u_{x+1} - u_x).$$

Modern work, especially that of Jordan (20) and Aitken (18), (19), has much facilitated the use of interpolation, without the aid of differences, in cases for which linear interpolation is thought not to be sufficiently accurate. A very useful improvement on the linear interpolate, based on four instead of two adjacent tabular values, is obtained by supplementing the linear interpolate from the two nearest entries by that from the next two which enclose them. This latter is

$$u'' = \tfrac{1}{3}\{(1+\theta)u_{x+2}+(1+\phi)u_{x-1}\}.$$

Curvature of the entries is shown by $u'$ differing from $u''$, the error of the former being generally of the same sign as that of the latter, though smaller. To complete the adjustment calculate the product $\theta\phi$, and use

$$u = \tfrac{1}{2}\{(2+\theta\phi)u'-\theta\phi u''\}$$

which is the correct 4-point interpolate, obtained without the use of differences.

In the tables in which mean proportional parts are given for each line of the table, only the proportional parts for 1 to 5 units are included, instead of, as is customary in tables of this type, for 1 to 9 units, it being intended that subtraction from the next higher tabular value should be used when interpolating for fractions above 0·5. The exact procedure will be clear from the examples below. The advantages of this innovation are two-fold : firstly the differences which have to be dealt with are smaller, thus facilitating the arithmetic (which should be done mentally), and secondly the errors arising from the use of *mean* proportional parts for the whole of each line are reduced.

The actual tables in which this type of arrangement has been adopted, for the whole or the greater part of the table, comprise : II (ordinates of the normal distribution), IX (probits), XXV and XXVI (logarithms), XXVIII (square roots), XXXI (sines), and XXXII (tangents). In tables XXV and XXVI (logarithms) the common practice is followed of tabulating the values for the numbers 10-19 on 20 lines instead of 10, so that five values instead of ten occur on each line, thus reducing the range covered by each set of mean proportional parts (which are here changing rather rapidly) to five units of the interval of tabulation.

In Table XXIX (reciprocals) mean differences of consecutive tabular values, instead of mean proportional parts, are given for each line of the table (thus allowing space for the tabulation of six figures instead of five). These mean differences correspond to ten times the mean proportional parts for 1 unit given in the other tables. This course has also been followed for Table VII (transformation of $r$ to $z$).

In those parts of Table XI (probits) and Table XXIX (reciprocals) where the first differences are changing so rapidly that the mean proportional parts or mean differences are too inaccurate to be of value, the actual first differences between each tabular value and the next are given.

In Table XXVII (squares) no differences are given. Squares of four-figure numbers may be easily derived by the use of the expansion of $(a+b)^2$, as indicated in the note below the table.

In all the above tables, except the last two lines of the table of probits and the first 20 lines (1·0-2·9) of the table of reciprocals, linear interpolation will give results accurate to one unit in the last tabulated place. The use of mean proportional parts or mean differences will, of course, give errors greater than this in those parts of the tables where the differences are changing rapidly. The actual magnitude of the error may be assessed by seeing how rapidly the proportional parts are changing from line to line, as illustrated in Example 13 below. The mean proportional parts should therefore be used with discretion when full tabular accuracy is required.

### *Example* 13

Find the natural logarithm of 11·06. We have $\log_e$ 11·1 = 2·40695. The proportional part for 4 is 357. Thus $\log_e$ 11·06 = 2·40338. The true value = 2·40333,5, the discrepancy being due to the use of the mean proportional part instead of the correct proportional part, which is 0·4 (2·40695−2·39790), *i.e.* 362. As will be seen by inspection the proportional parts are here changing rather rapidly. A closer approximation would have been obtained by taking this change into account, interpolating roughly in the required column of proportional parts, *i.e.* between 357 and 374.

### *Example* 14

Find the square root of 29·426. The square root of 29·4 is 5·4222, and since 2·6 = 3−0·4 the required proportional part is 28−3·7 or 24, giving the value 5·4246. (True value = 5·42457.)

### *Example* 15

Find the angle whose tangent is 0·79134. The nearest tabular value, 0·79259, is that for 38° 24′. The difference is −125. To the nearest minute, therefore, the angle is 38° 21′, a closer approximation being 38° 21·4′. (True value = 38° 21·36′.)

### *Example* 16

Find the angle whose tangent is 2·82. Ordinary inverse interpolation is here impossible, since $(90^\circ - \theta)\tan\theta$ is tabulated, but any approximation may be rapidly improved by dividing the tabular value by the given tangent, and subtracting from 90°. First find a rough approximation by interpolation in the auxiliary marginal table of tangents. This gives a value between 70·4° and 70·5°, *i.e.* between 70° 24′ and 70° 30′.

We then obtain

| Angle. | Tabular Value ($x$). | $x/2{\cdot}82$. | *Plus* Original Angle. |
|---|---|---|---|
| 70·4 | 55·043 | 19·5188 | 89·9188 |
| 70·5 | 55·066 | 19·5270 | 90·0270 |

The second column gives the tabular values from the table, the third is the result of dividing these by the given tangent 2·82. Subtracting either of these from 90° gives a better approximation. For full accuracy we use the fourth column, which is the sum of these quotients and the original angles. Inverse linear interpolation between the last two values for the value 90 will now give the required angle, namely 70·4750, *i.e.* 70° 28.50′. (True value = 70° 28·499′.)

# REFERENCES

### *Books and Monographs*

1. R. A. Fisher (1950) . . . Statistical methods for research workers (11th edition). Oliver and Boyd, Edinburgh.

2. R. A. Fisher (1949) . . . The design of experiments (5th edition). Oliver and Boyd, Edinburgh.

3. F. Yates (1937) . . . The design and analysis of factorial experiments. Imperial Bureau of Soil Science, Harpenden.

3a. D. J. Finney (1951) . . . Probit analysis (2nd edition). Cambridge University Press.

3b. W. G. Cochran and G. M. Cox (1950) Experimental designs. Wiley, New York.

### *Papers*

4. C. I. Bliss (1935) . . . The calculation of the dosage-mortality curve. Annals of Applied Biology, xxii, 134-167.

5. C. I. Bliss (1935) . . . The comparison of dosage-mortality data. Annals of Applied Biology, xxii, 307-333.

6. C. I. Bliss (1937) . . . The calculation of the time-mortality curve. Annals of Applied Biology, xxiv, 815-852.

7. C. I. Bliss (1938) . . . The determination of the dosage-mortality curve from small numbers. Quarterly Journal of Pharmacy and Pharmacology, xi, 192-216.

8. W. G. Cochran (1938) . . Some difficulties in the statistical analysis of replicated experiments. Empire Journal of Experimental Agriculture, vi, 157-175.

9. C. G. Colcord and L. S. Deming (1935) The one-tenth per cent. level of "*z*." Sankhyā, ii, 423-424.

10. F. A. E. Crew (1932) . . . Inheritance of educability. Proceedings of the Sixth International Congress of Genetics, i, 121-134.

11. R. A. Fisher and F. Yates (1934) The six by six Latin squares. Proceedings of the Cambridge Philosophical Society, xxx, 492-507.

12. W. L. Stevens (1937) . . Significance of grouping. Annals of Eugenics, viii, 57-69.

13. W. L. Stevens (1937) . . A test for uniovular twins in mice. Annals of Eugenics, viii, 70-73.

14. F. Yates (1934) . . . Contingency tables involving small numbers and the $\chi^2$ test. Journal of the Royal Statistical Society, Supplement, i, 217-235.

15. F. Yates (1936) . . . A new method of arranging variety trials involving a large number of varieties. Journal of Agricultural Science, xxvi, 424-455.

16. F. Yates (1936) . . . Incomplete randomised blocks. Annals of Eugenics, vii, 121-140.

17. F. Yates (1937) . . . A further note on the arrangement of variety trials: quasi-Latin squares. Annals of Eugenics, vii, 319-332.

18. A. C. Aitken (1932) . . . On interpolation by iteration of proportional parts, without the use of differences. Proceedings of the Edinburgh Mathematical Society, Series 2, iii, 56-76.

19. A. C. Aitken, G. J. Lidstone, and J. Wishart (1932) Interpolation without printed differences: Jordan's and Aitken's formulæ. Mathematical Gazette, xvi, 14-25.

20. C. Jordan (1932) . . . Sur une formule d'interpolation. Atti del Congresso Internazionale dei Matematici, Bologna, vi, 157-177.

21. W. J. Youden (1937) . . . Use of incomplete block replications in estimating tobacco-mosaic virus. Contributions from Boyce Thompson Institute, ix, 41-48.

22. H. W. Norton (1939) . . The 7×7 squares. Annals of Eugenics, ix, 269-307.

23. F. Yates (1940) . . . The recovery of inter-block information in balanced incomplete block designs. Annals of Eugenics, x, 317-325.

24. W. L. Stevens (1942) . . Accuracy of mutation rates. Journal of Genetics, xliii, 301-307.

25. F. Yates (1939) . . . The recovery of inter-block information in variety trials arranged in three dimensional lattices. Annals of Eugenics, ix, 136-156.

26. G. M. Cox, R. C. Eckhardt and W. G. Cochran (1940) The analysis of lattice and triple lattice experiments in corn varietal tests. Iowa Agricultural Experiment Station Research Bulletin, 281.

27. F. YATES (1940) . . . Lattice squares. Journal of Agricultural Science, xxx, 672-687.

28. R. C. BOSE and K. R. NAIR (1939) Partially balanced incomplete block designs. Sankhyā, iv, 337-372.

29. F. YATES (1940) . . . Modern experimental design and its function in plant selection. Empire Journal of Experimental Agriculture, viii, 223-230.

30. R. A. FISHER (1921) . . . On the mathematical foundations of theoretical statistics. Philosophical Transactions of the Royal Society of London, ccxxii, 309-368.

31. R. C. BOSE (1939) . . . On the construction of balanced incomplete block designs. Annals of Eugenics, ix, 353-399.

32. K. N. BHATTACHARYA (1944) . On a new symmetrical balanced incomplete block design. Bulletin of the Calcutta Mathematical Society, xxxvi, 91-96.

33. W. U. BEHRENS (1929) . . Ein Beitrag zur Fehlerberechnung bei wenigen Beobachtungen. Landwirtschaftliche Jahrbücher, lxviii, 807-837.

34. R. A. FISHER (1939) . . . The comparison of samples with possibly unequal variances. Annals of Eugenics, ix, 174-180.

35. R. A. FISHER (1941) . . . The asymptotic approach to Behrens's integral, with further tables for the *d* test of significance. Annals of Eugenics, ii, 141-172.

36. P. V. SUKHATME (1938) . . On Fisher and Behrens's test of significance for the difference in means of two normal samples. Sankhyā, iv, 39-48.

37. D. J. FINNEY (1944) . . . The application of the probit method to toxicity test data adjusted for mortality in the controls. Annals of Applied Biology, xxxi, 68-74.

38. R. L. ANDERSON and E. E. HOUSEMAN (1942) Tables of orthogonal polynomial values extended to N = 104. Research Bulletin 292, Ames, Iowa.

39. D. VAN DER REYDEN (1943) . Curve fitting by the orthogonal polynomials of least squares. Onderstepoort Journal of Veterinary Science and Animal Industry, xxviii, 355-404.

40. C. M. THOMPSON (1941) . . Tables of percentage points of the incomplete Beta function. Biometrika, xxxii, 151-181.

41. M. MERRINGTON and C. M. THOMPSON (1943) Tables of percentage points of the inverted Beta (F) distribution. Biometrika, xxxiii, 73-88.

42. V. G. PANSE and G. R. AYACHIT (1944) Ten per cent. probability of *z* and the variance ratio. Indian Journal of Agricultural Science, xiv, 244-247.

43. J. T. MARTIN (1940) . . . The problem of the evaluation of rotenone-containing plants. V. The relative toxicities of different species of derris. Annals of Applied Biology, xxvii, 274-294.

44. Q. M. HUSSAIN (1946) . . Impossibility of the symmetrical incomplete block design with $\lambda = 2$, $k = 7$. Sankhyā, vii, 317-322.

45. Q. M. HUSSAIN (1945) . . Symmetrical incomplete block designs with $\lambda = 2$, $k = 8$ or 9. Bulletin of the Calcutta Mathematical Society, xxxvii, 115-123.

46. H. K. NANDI (1945) . . . On the relation between certain types of tactical configurations. Bulletin of the Calcutta Mathematical Society, xxxvii, 92-94.

47. C. R. RAO (1946) . . . Difference sets and combinatorial arrangements derivable from finite geometrics. Proceedings of the National Institute of Sciences of India, xii, 123-135.

48. M. P. SCHÜTZENBERGER (1949) . A non-existence theorem for an infinite family of symmetrical block designs. Annals of Eugenics, xiv, 286-287.

49. S. S. SHRIKHANDE (1950) . . The impossibility of certain symmetrical balanced incomplete block designs. Annals of Mathematical Statistics, xxi, 106-111.

50. K. N. BHATTACHARYA (1946) . A new solution in symmetrical balanced incomplete block designs. Sankhyā, vii, 423-424.

51. W. S. CONNOR (1951) . . . The structure of balanced incomplete block designs, and the impossibility of certain unsymmetrical cases. (Abstract) Annals of Mathematical Statistics, xxii, 312-313.

52. C. A. B. SMITH and H. O. HARTLEY (1948) The construction of Youden squares. Journal of the Royal Statistical Society, B, x, 262-263.

53. S. CHOWLA and H. J. RYSER (1950) Combinatorial problems. Canadian Journal of Mathematics, ii, 93-99.

54. S. S. SHRIKHANDE (1951) . . Designs for two-way elimination of heterogeneity. Annals of Mathematical Statistics, xxii, 235-247.

55. R. A. FISHER (1929) . . . Tests of significance in harmonic analysis. Proceedings of the Royal Society of London, Series A, cxxv, 54-59.

## REFERENCES—*continued*

56. R. A. Fisher (1950) Gene frequencies in a cline determined by selection and diffusion. Biometrics, vi, 353-361.

57. D. J. Finney (1949) The estimation of the frequency of recombinations. I. Matings of known phase. Journal of Genetics, xlix, 159-176.

58. R. A. Fisher, M. F. Lyon and A. R. G. Owen (1947) The sex chromosome in the house mouse. Heredity, i, 335-365.

59. R. A. Fisher (1948) A quantitative theory of genetic recombination and chiasma formation. Biometrics, iv, 1-13.

60. A. R. G. Owen (1949) The theory of genetical recombination. I. Long-chromosome arms. Proceedings of the Royal Society of London, Series B, cxxxvi, 67-94.

61. A. R. G. Owen (1950) The theory of genetical recombination. Advances in Genetics, iii, 117-157.

62. K. Mather (1949) The analysis of extinction time data in bioassay. Biometrics, v, 127-143.

63. R. C. Bose (1938) On the application of the properties of Galois fields to the problem of construction of Hyper-Græco-Latin squares. Sankhyā, iii, 323-338.

64. W. L. Stevens (1939) The completely orthogonalized Latin square. Annals of Eugenics, ix, 82-93.

65. R. C. Bose and K. R. Nair (1941) On complete sets of Latin squares. Sankhyā, v, 361-382.

66. A. Sade (1951) An omission in Norton's list of 7×7 squares. Annals of Mathematical Statistics, xxii, 306-307.

## TABLE I. THE NORMAL DISTRIBUTION

| P | ·00 | ·01 | ·02 | ·03 | ·04 | ·05 | ·06 | ·07 | ·08 | ·09 |
|---|---|---|---|---|---|---|---|---|---|---|
| ·0 | ∞ | 2·575829 | 2·326348 | 2·170090 | 2·053749 | 1·959964 | 1·880794 | 1·811911 | 1·750686 | 1·695398 |
| ·1 | 1·644854 | 1·598193 | 1·554774 | 1·514102 | 1·475791 | 1·439531 | 1·405072 | 1·372204 | 1·340755 | 1·310579 |
| ·2 | 1·281552 | 1·253565 | 1·226528 | 1·200359 | 1·174987 | 1·150349 | 1·126391 | 1·103063 | 1·080319 | 1·058122 |
| ·3 | 1·036433 | 1·015222 | ·994458 | ·974114 | ·954165 | ·934589 | ·915365 | ·896473 | ·877896 | ·859617 |
| ·4 | ·841621 | ·823894 | ·806421 | ·789192 | ·772193 | ·755415 | ·738847 | ·722479 | ·706303 | ·690309 |
| ·5 | ·674490 | ·658838 | ·643345 | ·628006 | ·612813 | ·597760 | ·582842 | ·568051 | ·553385 | ·538836 |
| ·6 | ·524401 | ·510073 | ·495850 | ·481727 | ·467699 | ·453762 | ·439913 | ·426148 | ·412463 | ·398855 |
| ·7 | ·385320 | ·371856 | ·358459 | ·345126 | ·331853 | ·318639 | ·305481 | ·292375 | ·279319 | ·266311 |
| ·8 | ·253347 | ·240426 | ·227545 | ·214702 | ·201893 | ·189118 | ·176374 | ·163658 | ·150969 | ·138304 |
| ·9 | ·125661 | ·113039 | ·100434 | ·087845 | ·075270 | ·062707 | ·050154 | ·037608 | ·025069 | ·012533 |

| P . . . | ·001 | ·000,1 | ·000,01 | ·000,001 | ·000,000,1 | ·000,000,01 | ·000,000,001 |
|---|---|---|---|---|---|---|---|
| $x$ . . . | 3·29053 | 3·89059 | 4·41717 | 4·89164 | 5·32672 | 5·73073 | 6·10941 |

The value of P for each entry is found by adding the column heading to the value in the left-hand margin. The corresponding value of $x$ is the deviation such that the probability of an observation falling outside the range from $-x$ to $+x$ is P. For example, P = ·03 for $x$ = 2·170090; so that 3 per cent. of normally distributed values will have positive or negative deviations exceeding the standard deviation in the ratio 2·170090 at least.

The table of probits (Table IX) provides a more extensive (P, $x$) table. The probit table refers to a single tail of the distribution, and the P's derived from that table must therefore be multiplied by 2 to bring them into line with the P's of Table I.

## TABLE II. ORDINATES OF THE NORMAL DISTRIBUTION

| $x$ | ·00 | ·01 | ·02 | ·03 | ·04 | ·05 | ·06 | ·07 | ·08 | ·09 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·0 | ·3989 | ·3989 | ·3989 | ·3988 | ·3986 | ·3984 | ·3982 | ·3980 | ·3977 | ·3973 | 0 | 0 | −1 | −1 | −1 |
| 0·1 | ·3970 | ·3965 | ·3961 | ·3956 | ·3951 | ·3945 | ·3939 | ·3932 | ·3925 | ·3918 | −1 | −1 | −2 | −2 | −3 |
| 0·2 | ·3910 | ·3902 | ·3894 | ·3885 | ·3876 | ·3867 | ·3857 | ·3847 | ·3836 | ·3825 | −1 | −2 | −3 | −4 | −5 |
| 0·3 | ·3814 | ·3802 | ·3790 | ·3778 | ·3765 | ·3752 | ·3739 | ·3725 | ·3712 | ·3697 | −1 | −3 | −4 | −5 | −6 |
| 0·4 | ·3683 | ·3668 | ·3653 | ·3637 | ·3621 | ·3605 | ·3589 | ·3572 | ·3555 | ·3538 | −2 | −3 | −5 | −6 | −8 |
| 0·5 | ·3521 | ·3503 | ·3485 | ·3467 | ·3448 | ·3429 | ·3410 | ·3391 | ·3372 | ·3352 | −2 | −4 | −6 | −8 | −9 |
| 0·6 | ·3332 | ·3312 | ·3292 | ·3271 | ·3251 | ·3230 | ·3209 | ·3187 | ·3166 | ·3144 | −2 | −4 | −6 | −8 | −10 |
| 0·7 | ·3123 | ·3101 | ·3079 | ·3056 | ·3034 | ·3011 | ·2989 | ·2966 | ·2943 | ·2920 | −2 | −5 | −7 | −9 | −11 |
| 0·8 | ·2897 | ·2874 | ·2850 | ·2827 | ·2803 | ·2780 | ·2756 | ·2732 | ·2709 | ·2685 | −2 | −5 | −7 | −9 | −12 |
| 0·9 | ·2661 | ·2637 | ·2613 | ·2589 | ·2565 | ·2541 | ·2516 | ·2492 | ·2468 | ·2444 | −2 | −5 | −7 | −10 | −12 |
| 1·0 | ·2420 | ·2396 | ·2371 | ·2347 | ·2323 | ·2299 | ·2275 | ·2251 | ·2227 | ·2203 | −2 | −5 | −7 | −10 | −12 |
| 1·1 | ·2179 | ·2155 | ·2131 | ·2107 | ·2083 | ·2059 | ·2036 | ·2012 | ·1989 | ·1965 | −2 | −5 | −7 | −10 | −12 |
| 1·2 | ·1942 | ·1919 | ·1895 | ·1872 | ·1849 | ·1826 | ·1804 | ·1781 | ·1758 | ·1736 | −2 | −5 | −7 | −9 | −11 |
| 1·3 | ·1714 | ·1691 | ·1669 | ·1647 | ·1626 | ·1604 | ·1582 | ·1561 | ·1539 | ·1518 | −2 | −4 | −7 | −9 | −11 |
| 1·4 | ·1497 | ·1476 | ·1456 | ·1435 | ·1415 | ·1394 | ·1374 | ·1354 | ·1334 | ·1315 | −2 | −4 | −6 | −8 | −10 |
| 1·5 | ·1295 | ·1276 | ·1257 | ·1238 | ·1219 | ·1200 | ·1182 | ·1163 | ·1145 | ·1127 | −2 | −4 | −6 | −7 | −9 |
| 1·6 | ·1109 | ·1092 | ·1074 | ·1057 | ·1040 | ·1023 | ·1006 | ·0989 | ·0973 | ·0957 | −2 | −3 | −5 | −7 | −8 |
| 1·7 | ·0940 | ·0925 | ·0909 | ·0893 | ·0878 | ·0863 | ·0848 | ·0833 | ·0818 | ·0804 | −2 | −3 | −5 | −6 | −8 |
| 1·8 | ·0790 | ·0775 | ·0761 | ·0748 | ·0734 | ·0721 | ·0707 | ·0694 | ·0681 | ·0669 | −1 | −3 | −4 | −5 | −7 |
| 1·9 | ·0656 | ·0644 | ·0632 | ·0620 | ·0608 | ·0596 | ·0584 | ·0573 | ·0562 | ·0551 | −1 | −2 | −4 | −5 | −6 |
| 2·0 | ·0540 | ·0529 | ·0519 | ·0508 | ·0498 | ·0488 | ·0478 | ·0468 | ·0459 | ·0449 | −1 | −2 | −3 | −4 | −5 |
| 2·1 | ·0440 | ·0431 | ·0422 | ·0413 | ·0404 | ·0396 | ·0387 | ·0379 | ·0371 | ·0363 | −1 | −2 | −3 | −3 | −4 |
| 2·2 | ·0355 | ·0347 | ·0339 | ·0332 | ·0325 | ·0317 | ·0310 | ·0303 | ·0297 | ·0290 | −1 | −1 | −2 | −3 | −4 |
| 2·3 | ·0283 | ·0277 | ·0270 | ·0264 | ·0258 | ·0252 | ·0246 | ·0241 | ·0235 | ·0229 | −1 | −1 | −2 | −2 | −3 |
| 2·4 | ·0224 | ·0219 | ·0213 | ·0208 | ·0203 | ·0198 | ·0194 | ·0189 | ·0184 | ·0180 | 0 | −1 | −1 | −2 | −2 |
| 2·5 | ·0175 | ·0171 | ·0167 | ·0163 | ·0158 | ·0154 | ·0151 | ·0147 | ·0143 | ·0139 | 0 | −1 | −1 | −2 | −2 |
| 2·6 | ·0136 | ·0132 | ·0129 | ·0126 | ·0122 | ·0119 | ·0116 | ·0113 | ·0110 | ·0107 | 0 | −1 | −1 | −1 | −2 |
| 2·7 | ·0104 | ·0101 | ·0099 | ·0096 | ·0093 | ·0091 | ·0088 | ·0086 | ·0084 | ·0081 | 0 | −1 | −1 | −1 | −1 |
| 2·8 | ·0079 | ·0077 | ·0075 | ·0073 | ·0071 | ·0069 | ·0067 | ·0065 | ·0063 | ·0061 | 0 | 0 | −1 | −1 | −1 |
| 2·9 | ·0060 | ·0058 | ·0056 | ·0055 | ·0053 | ·0051 | ·0050 | ·0048 | ·0047 | ·0046 | 0 | 0 | 0 | −1 | −1 |
|  | ·0 | ·1 | ·2 | ·3 | ·4 | ·5 | ·6 | ·7 | ·8 | ·9 |  |  |  |  |  |
| 3·0 | ·0044 | ·0033 | ·0024 | ·0017 | ·0012 | ·0009 | ·0006 | ·0004 | ·0003 | ·0002 |  |  |  |  |  |

# TABLE III. DISTRIBUTION OF *t*

Probability.

| *n* | ·9 | ·8 | ·7 | ·6 | ·5 | ·4 | ·3 | ·2 | ·1 | ·05 | ·02 | ·01 | ·001 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | ·158 | ·325 | ·510 | ·727 | 1·000 | 1·376 | 1·963 | 3·078 | 6·314 | 12·706 | 31·821 | 63·657 | 636·619 |
| 2 | ·142 | ·289 | ·445 | ·617 | ·816 | 1·061 | 1·386 | 1·886 | 2·920 | 4·303 | 6·965 | 9·925 | 31·598 |
| 3 | ·137 | ·277 | ·424 | ·584 | ·765 | ·978 | 1·250 | 1·638 | 2·353 | 3·182 | 4·541 | 5·841 | 12·924 |
| 4 | ·134 | ·271 | ·414 | ·569 | ·741 | ·941 | 1·190 | 1·533 | 2·132 | 2·776 | 3·747 | 4·604 | 8·610 |
| 5 | ·132 | ·267 | ·408 | ·559 | ·727 | ·920 | 1·156 | 1·476 | 2·015 | ✓2·571 | 3·365 | 4·032 | 6·869 |
| 6 | ·131 | ·265 | ·404 | ·553 | ·718 | ·906 | 1·134 | 1·440 | 1·943 | 2·447 | 3·143 | 3·707 | 5·959 |
| 7 | ·130 | ·263 | ·402 | ·549 | ·711 | ·896 | 1·119 | 1·415 | 1·895 | 2·365 | 2·998 | 3·499 | 5·408 |
| 8 | ·130 | ·262 | ·399 | ·546 | ·706 | ·889 | 1·108 | 1·397 | 1·860 | 2·306 | 2·896 | 3·355 | 5·041 |
| 9 | ·129 | ·261 | ·398 | ·543 | ·703 | ·883 | 1·100 | 1·383 | 1·833 | 2·262 | 2·821 | 3·250 | 4·781 |
| 10 | ·129 | ·260 | ·397 | ·542 | ·700 | ·879 | 1·093 | 1·372 | 1·812 | 2·228 | 2·764 | 3·169 | 4·587 |
| 11 | ·129 | ·260 | ·396 | ·540 | ·697 | ·876 | 1·088 | 1·363 | 1·796 | 2·201 | 2·718 | 3·106 | 4·437 |
| 12 | ·128 | ·259 | ·395 | ·539 | ·695 | ·873 | 1·083 | 1·356 | 1·782 | 2·179 | 2·681 | 3·055 | 4·318 |
| 13 | ·128 | ·259 | ·394 | ·538 | ·694 | ·870 | 1·079 | 1·350 | 1·771 | 2·160 | 2·650 | 3·012 | 4·221 |
| 14 | ·128 | ·258 | ·393 | ·537 | ·692 | ·868 | 1·076 | 1·345 | 1·761 | 2·145 | 2·624 | 2·977 | 4·140 |
| 15 | ·128 | ·258 | ·393 | ·536 | ·691 | ·866 | 1·074 | 1·341 | 1·753 | 2·131 | 2·602 | 2·947 | 4·073 |
| 16 | ·128 | ·258 | ·392 | ·535 | ·690 | ·865 | 1·071 | 1·337 | 1·746 | 2·120 | 2·583 | 2·921 | 4·015 |
| 17 | ·128 | ·257 | ·392 | ·534 | ·689 | ·863 | 1·069 | 1·333 | 1·740 | 2·110 | 2·567 | 2·898 | 3·965 |
| 18 | ·127 | ·257 | ·392 | ·534 | ·688 | ·862 | 1·067 | 1·330 | 1·734 | 2·101 | 2·552 | 2·878 | 3·922 |
| 19 | ·127 | ·257 | ·391 | ·533 | ·688 | ·861 | 1·066 | 1·328 | 1·729 | 2·093 | 2·539 | 2·861 | 3·883 |
| 20 | ·127 | ·257 | ·391 | ·533 | ·687 | ·860 | 1·064 | 1·325 | 1·725 | 2·086 | 2·528 | 2·845 | 3·850 |
| 21 | ·127 | ·257 | ·391 | ·532 | ·686 | ·859 | 1·063 | 1·323 | 1·721 | 2·080 | 2·518 | 2·831 | 3·819 |
| 22 | ·127 | ·256 | ·390 | ·532 | ·686 | ·858 | 1·061 | 1·321 | 1·717 | 2·074 | 2·508 | 2·819 | 3·792 |
| 23 | ·127 | ·256 | ·390 | ·532 | ·685 | ·858 | 1·060 | 1·319 | 1·714 | 2·069 | 2·500 | 2·807 | 3·767 |
| 24 | ·127 | ·256 | ·390 | ·531 | ·685 | ·857 | 1·059 | 1·318 | 1·711 | 2·064 | 2·492 | 2·797 | 3·745 |
| 25 | ·127 | ·256 | ·390 | ·531 | ·684 | ·856 | 1·058 | 1·316 | 1·708 | 2·060 | 2·485 | 2·787 | 3·725 |
| 26 | ·127 | ·256 | ·390 | ·531 | ·684 | ·856 | 1·058 | 1·315 | 1·706 | 2·056 | 2·479 | 2·779 | 3·707 |
| 27 | ·127 | ·256 | ·389 | ·531 | ·684 | ·855 | 1·057 | 1·314 | 1·703 | 2·052 | 2·473 | 2·771 | 3·690 |
| 28 | ·127 | ·256 | ·389 | ·530 | ·683 | ·855 | 1·056 | 1·313 | 1·701 | 2·048 | 2·467 | 2·763 | 3·674 |
| 29 | ·127 | ·256 | ·389 | ·530 | ·683 | ·854 | 1·055 | 1·311 | 1·699 | 2·045 | 2·462 | 2·756 | 3·659 |
| 30 | ·127 | ·256 | ·389 | ·530 | ·683 | ·854 | 1·055 | 1·310 | 1·697 | 2·042 | 2·457 | 2·750 | 3·646 |
| 40 | ·126 | ·255 | ·388 | ·529 | ·681 | ·851 | 1·050 | 1·303 | 1·684 | 2·021 | 2·423 | 2·704 | 3·551 |
| 60 | ·126 | ·254 | ·387 | ·527 | ·679 | ·848 | 1·046 | 1·296 | 1·671 | 2·000 | 2·390 | 2·660 | 3·460 |
| 120 | ·126 | ·254 | ·386 | ·526 | ·677 | ·845 | 1·041 | 1·289 | 1·658 | 1·980 | 2·358 | 2·617 | 3·373 |
| ∞ | ·126 | ·253 | ·385 | ·524 | ·674 | ·842 | 1·036 | 1·282 | 1·645 | 1·960 | 2·326 | 2·576 | 3·291 |

# TABLE IV. DISTRIBUTION OF $\chi^2$

Probability.

| $n$ | ·99 | ·98 | ·95 | ·90 | ·80 | ·70 | ·50 | ·30 | ·20 | ·10 | ·05 | ·02 | ·01 | ·001 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | ·0$^{3}$157 | ·0$^{3}$628 | ·00393 | ·0158 | ·0642 | ·148 | ·455 | 1·074 | 1·642 | 2·706 | 3·841 | 5·412 | 6·635 | 10·827 |
| 2 | ·0201 | ·0404 | ·103 | ·211 | ·446 | ·713 | 1·386 | 2·408 | 3·219 | 4·605 | 5·991 | 7·824 | 9·210 | 13·815 |
| 3 | ·115 | ·185 | ·352 | ·584 | 1·005 | 1·424 | 2·366 | 3·665 | 4·642 | 6·251 | 7·815 | 9·837 | 11·345 | 16·266 |
| 4 | ·297 | ·429 | ·711 | 1·064 | 1·649 | 2·195 | 3·357 | 4·878 | 5·989 | 7·779 | 9·488 | 11·668 | 13·277 | 18·467 |
| 5 | ·554 | ·752 | 1·145 | 1·610 | 2·343 | 3·000 | 4·351 | 6·064 | 7·289 | 9·236 | 11·070 | 13·388 | 15·086 | 20·515 |
| 6 | ·872 | 1·134 | 1·635 | 2·204 | 3·070 | 3·828 | 5·348 | 7·231 | 8·558 | 10·645 | 12·592 | 15·033 | 16·812 | 22·457 |
| 7 | 1·239 | 1·564 | 2·167 | 2·833 | 3·822 | 4·671 | 6·346 | 8·383 | 9·803 | 12·017 | 14·067 | 16·622 | 18·475 | 24·322 |
| 8 | 1·646 | 2·032 | 2·733 | 3·490 | 4·594 | 5·527 | 7·344 | 9·524 | 11·030 | 13·362 | 15·507 | 18·168 | 20·090 | 26·125 |
| 9 | 2·088 | 2·532 | 3·325 | 4·168 | 5·380 | 6·393 | 8·343 | 10·656 | 12·242 | 14·684 | 16·919 | 19·679 | 21·666 | 27·877 |
| 10 | 2·558 | 3·059 | 3·940 | 4·865 | 6·179 | 7·267 | 9·342 | 11·781 | 13·442 | 15·987 | 18·307 | 21·161 | 23·209 | 29·588 |
| 11 | 3·053 | 3·609 | 4·575 | 5·578 | 6·989 | 8·148 | 10·341 | 12·899 | 14·631 | 17·275 | 19·675 | 22·618 | 24·725 | 31·264 |
| 12 | 3·571 | 4·178 | 5·226 | 6·304 | 7·807 | 9·034 | 11·340 | 14·011 | 15·812 | 18·549 | 21·026 | 24·054 | 26·217 | 32·909 |
| 13 | 4·107 | 4·765 | 5·892 | 7·042 | 8·634 | 9·926 | 12·340 | 15·119 | 16·985 | 19·812 | 22·362 | 25·472 | 27·688 | 34·528 |
| 14 | 4·660 | 5·368 | 6·571 | 7·790 | 9·467 | 10·821 | 13·339 | 16·222 | 18·151 | 21·064 | 23·685 | 26·873 | 29·141 | 36·123 |
| 15 | 5·229 | 5·985 | 7·261 | 8·547 | 10·307 | 11·721 | 14·339 | 17·322 | 19·311 | 22·307 | 24·996 | 28·259 | 30·578 | 37·697 |
| 16 | 5·812 | 6·614 | 7·962 | 9·312 | 11·152 | 12·624 | 15·338 | 18·418 | 20·465 | 23·542 | 26·296 | 29·633 | 32·000 | 39·252 |
| 17 | 6·408 | 7·255 | 8·672 | 10·085 | 12·002 | 13·531 | 16·338 | 19·511 | 21·615 | 24·769 | 27·587 | 30·995 | 33·409 | 40·790 |
| 18 | 7·015 | 7·906 | 9·390 | 10·865 | 12·857 | 14·440 | 17·338 | 20·601 | 22·760 | 25·989 | 28·869 | 32·346 | 34·805 | 42·312 |
| 19 | 7·633 | 8·567 | 10·117 | 11·651 | 13·716 | 15·352 | 18·338 | 21·689 | 23·900 | 27·204 | 30·144 | 33·687 | 36·191 | 43·820 |
| 20 | 8·260 | 9·237 | 10·851 | 12·443 | 14·578 | 16·266 | 19·337 | 22·775 | 25·038 | 28·412 | 31·410 | 35·020 | 37·566 | 45·315 |
| 21 | 8·897 | 9·915 | 11·591 | 13·240 | 15·445 | 17·182 | 20·337 | 23·858 | 26·171 | 29·615 | 32·671 | 36·343 | 38·932 | 46·797 |
| 22 | 9·542 | 10·600 | 12·338 | 14·041 | 16·314 | 18·101 | 21·337 | 24·939 | 27·301 | 30·813 | 33·924 | 37·659 | 40·289 | 48·268 |
| 23 | 10·196 | 11·293 | 13·091 | 14·848 | 17·187 | 19·021 | 22·337 | 26·018 | 28·429 | 32·007 | 35·172 | 38·968 | 41.638 | 49·728 |
| 24 | 10·856 | 11·992 | 13·848 | 15·659 | 18·062 | 19·943 | 23·337 | 27·096 | 29·553 | 33·196 | 36·415 | 40·270 | 42·980 | 51·179 |
| 25 | 11·524 | 12·697 | 14·611 | 16·473 | 18·940 | 20·867 | 24·337 | 28·172 | 30·675 | 34·382 | 37·652 | 41·566 | 44·314 | 52·620 |
| 26 | 12·198 | 13·409 | 15·379 | 17·292 | 19·820 | 21·792 | 25·336 | 29·246 | 31·795 | 35·563 | 38·885 | 42·856 | 45·642 | 54·052 |
| 27 | 12·879 | 14·125 | 16·151 | 18·114 | 20·703 | 22·719 | 26·336 | 30·319 | 32·912 | 36·741 | 40·113 | 44·140 | 46·963 | 55·476 |
| 28 | 13·565 | 14·847 | 16·928 | 18·939 | 21·588 | 23·647 | 27·336 | 31·391 | 34·027 | 37·916 | 41·337 | 45·419 | 48·278 | 56·893 |
| 29 | 14·256 | 15·574 | 17·708 | 19·768 | 22·475 | 24·577 | 28·336 | 32·461 | 35·139 | 39·087 | 42·557 | 46·693 | 49·588 | 58·302 |
| 30 | 14·953 | 16·306 | 18·493 | 20·599 | 23·364 | 25·508 | 29·336 | 33·530 | 36·250 | 40·256 | 43·773 | 47·962 | 50·892 | 59·703 |

For larger values of $n$, the expression $\sqrt{2\chi^2}-\sqrt{2n-1}$ may be used as a normal deviate with unit variance, remembering that the probability for $\chi^2$ corresponds with that of a single tail of the normal curve.

## TABLE V. DISTRIBUTION OF $z$

### 20 Per Cent. Points (H. W. Norton)

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1·1242 | 1·2425 | 1·2849 | 1·3067 | 1·3198 | 1·3286 | 1·3397 | 1·3508 | 1·3619 | 1·3730 |
| 2 | ·6343 | ·6931 | ·7125 | ·7219 | ·7275 | ·7313 | ·7360 | ·7406 | ·7453 | ·7500 |
| 3 | ·4933 | ·5299 | ·5385 | ·5418 | ·5435 | ·5444 | ·5453 | ·5461 | ·5465 | ·5467 |
| 4 | ·4274 | ·4526 | ·4551 | ·4547 | ·4537 | ·4528 | ·4512 | ·4491 | ·4464 | ·4431 |
| 5 | ·3892 | ·4075 | ·4062 | ·4032 | ·4004 | ·3982 | ·3947 | ·3905 | ·3853 | ·3791 |
| 6 | ·3645 | ·3780 | ·3740 | ·3692 | ·3651 | ·3618 | ·3569 | ·3510 | ·3438 | ·3350 |
| 7 | ·3471 | ·3573 | ·3512 | ·3450 | ·3399 | ·3358 | ·3298 | ·3224 | ·3135 | ·3025 |
| 8 | ·3342 | ·3419 | ·3342 | ·3269 | ·3211 | ·3163 | ·3093 | ·3008 | ·2904 | ·2774 |
| 9 | ·3243 | ·3300 | ·3211 | ·3130 | ·3064 | ·3011 | ·2933 | ·2838 | ·2721 | ·2573 |
| 10 | ·3164 | ·3206 | ·3107 | ·3018 | ·2946 | ·2889 | ·2805 | ·2701 | ·2572 | ·2407 |
| 11 | ·3100 | ·3129 | ·3021 | ·2926 | ·2850 | ·2790 | ·2699 | ·2588 | ·2449 | ·2268 |
| 12 | ·3047 | ·3065 | ·2951 | ·2850 | ·2770 | ·2706 | ·2611 | ·2493 | ·2345 | ·2149 |
| 13 | ·3002 | ·3011 | ·2891 | ·2786 | ·2703 | ·2636 | ·2536 | ·2412 | ·2255 | ·2046 |
| 14 | ·2964 | ·2965 | ·2840 | ·2732 | ·2644 | ·2575 | ·2471 | ·2342 | ·2178 | ·1956 |
| 15 | ·2931 | ·2926 | ·2795 | ·2684 | ·2594 | ·2522 | ·2415 | ·2281 | ·2110 | ·1876 |
| 16 | ·2903 | ·2891 | ·2757 | ·2642 | ·2550 | ·2476 | ·2366 | ·2228 | ·2050 | ·1805 |
| 17 | ·2877 | ·2860 | ·2723 | ·2605 | ·2511 | ·2436 | ·2322 | ·2180 | ·1997 | ·1741 |
| 18 | ·2855 | ·2833 | ·2692 | ·2572 | ·2476 | ·2399 | ·2283 | ·2138 | ·1949 | ·1682 |
| 19 | ·2835 | ·2809 | ·2665 | ·2543 | ·2445 | ·2367 | ·2249 | ·2100 | ·1905 | ·1629 |
| 20 | ·2817 | ·2787 | ·2641 | ·2517 | ·2418 | ·2338 | ·2217 | ·2066 | ·1866 | ·1581 |
| 21 | ·2800 | ·2767 | ·2619 | ·2493 | ·2392 | ·2311 | ·2189 | ·2034 | ·1831 | ·1536 |
| 22 | ·2786 | ·2750 | ·2599 | ·2471 | ·2369 | ·2287 | ·2163 | ·2006 | ·1798 | ·1495 |
| 23 | ·2772 | ·2733 | ·2581 | ·2452 | ·2348 | ·2265 | ·2139 | ·1980 | ·1767 | ·1457 |
| 24 | ·2760 | ·2719 | ·2564 | ·2433 | ·2329 | ·2245 | ·2117 | ·1956 | ·1740 | ·1421 |
| 25 | ·2749 | ·2705 | ·2549 | ·2417 | ·2311 | ·2226 | ·2097 | ·1933 | ·1715 | ·1388 |
| 26 | ·2738 | ·2692 | ·2534 | ·2401 | ·2295 | ·2209 | ·2079 | ·1913 | ·1691 | ·1357 |
| 27 | ·2729 | ·2681 | ·2521 | ·2387 | ·2280 | ·2193 | ·2062 | ·1894 | ·1669 | ·1328 |
| 28 | ·2720 | ·2670 | ·2509 | ·2374 | ·2266 | ·2178 | ·2046 | ·1876 | ·1648 | ·1300 |
| 29 | ·2711 | ·2660 | ·2498 | ·2362 | ·2253 | ·2165 | ·2031 | ·1860 | ·1629 | ·1274 |
| 30 | ·2703 | ·2650 | ·2487 | ·2350 | ·2241 | ·2152 | ·2017 | ·1844 | ·1611 | ·1250 |
| 40 | ·2647 | ·2582 | ·2411 | ·2267 | ·2152 | ·2058 | ·1915 | ·1731 | ·1477 | ·1062 |
| 60 | ·2592 | ·2514 | ·2334 | ·2184 | ·2063 | ·1965 | ·1813 | ·1616 | ·1338 | ·0848 |
| 120 | ·2536 | ·2446 | ·2258 | ·2101 | ·1974 | ·1870 | ·1710 | ·1499 | ·1193 | ·0582 |
| ∞ | ·2481 | ·2379 | ·2183 | ·2018 | ·1885 | ·1776 | ·1606 | ·1379 | ·1041 | 0 |

For high values of $n_1$ and $n_2$, $z$ (20 per cent.) $= \frac{0{\cdot}8416}{\sqrt{h-0{\cdot}6}} - 0{\cdot}4514\left(\frac{1}{n_1}-\frac{1}{n_2}\right)$ approximately, where $\frac{2}{h} = \frac{1}{n_1} + \frac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of $n_1$ and $n_2$ are taken.

## TABLE V. VARIANCE RATIO

### 20 Per Cent. Points of $e^{2z}$ (H. W. Norton)

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 9·47 | 12·00 | 13·06 | 13·64 | 14·01 | 14·26 | 14·58 | 14·90 | 15·24 | 15·58 |
| 2 | 3·56 | 4·00 | 4·16 | 4·24 | 4·28 | 4·32 | 4·36 | 4·40 | 4·44 | 4·48 |
| 3 | 2·68 | 2·89 | 2·94 | 2·96 | 2·97 | 2·97 | 2·98 | 2·98 | 2·98 | 2·98 |
| 4 | 2·35 | 2·47 | 2·48 | 2·48 | 2·48 | 2·47 | 2·47 | 2·46 | 2·44 | 2·43 |
| 5 | 2·18 | 2·26 | 2·25 | 2·24 | 2·23 | 2·22 | 2·20 | 2·18 | 2·16 | 2·13 |
| 6 | 2·07 | 2·13 | 2·11 | 2·09 | 2·08 | 2·06 | 2·04 | 2·02 | 1·99 | 1·95 |
| 7 | 2·00 | 2·04 | 2·02 | 1·99 | 1·97 | 1·96 | 1·93 | 1·91 | 1·87 | 1·83 |
| 8 | 1·95 | 1·98 | 1·95 | 1·92 | 1·90 | 1·88 | 1·86 | 1·83 | 1·79 | 1·74 |
| 9 | 1·91 | 1·94 | 1·90 | 1·87 | 1·85 | 1·83 | 1·80 | 1·76 | 1·72 | 1·67 |
| 10 | 1·88 | 1·90 | 1·86 | 1·83 | 1·80 | 1·78 | 1·75 | 1·72 | 1·67 | 1·62 |
| 11 | 1·86 | 1·87 | 1·83 | 1·80 | 1·77 | 1·75 | 1·72 | 1·68 | 1·63 | 1·57 |
| 12 | 1·84 | 1·85 | 1·80 | 1·77 | 1·74 | 1·72 | 1·69 | 1·65 | 1·60 | 1·54 |
| 13 | 1·82 | 1·83 | 1·78 | 1·75 | 1·72 | 1·69 | 1·66 | 1·62 | 1·57 | 1·51 |
| 14 | 1·81 | 1·81 | 1·76 | 1·73 | 1·70 | 1·67 | 1·64 | 1·60 | 1·55 | 1·48 |
| 15 | 1·80 | 1·79 | 1·75 | 1·71 | 1·68 | 1·66 | 1·62 | 1·58 | 1·53 | 1·46 |
| 16 | 1·79 | 1·78 | 1·74 | 1·70 | 1·67 | 1·64 | 1·61 | 1·56 | 1·51 | 1·43 |
| 17 | 1·78 | 1·77 | 1·72 | 1·68 | 1·65 | 1·63 | 1·59 | 1·55 | 1·49 | 1·42 |
| 18 | 1·77 | 1·76 | 1·71 | 1·67 | 1·64 | 1·62 | 1·58 | 1·53 | 1·48 | 1·40 |
| 19 | 1·76 | 1·75 | 1·70 | 1·66 | 1·63 | 1·61 | 1·57 | 1·52 | 1·46 | 1·39 |
| 20 | 1·76 | 1·75 | 1·70 | 1·65 | 1·62 | 1·60 | 1·56 | 1·51 | 1·45 | 1·37 |
| 21 | 1·75 | 1·74 | 1·69 | 1·65 | 1·61 | 1·59 | 1·55 | 1·50 | 1·44 | 1·36 |
| 22 | 1·75 | 1·73 | 1·68 | 1·64 | 1·61 | 1·58 | 1·54 | 1·49 | 1·43 | 1·35 |
| 23 | 1·74 | 1·73 | 1·68 | 1·63 | 1·60 | 1·57 | 1·53 | 1·49 | 1·42 | 1·34 |
| 24 | 1·74 | 1·72 | 1·67 | 1·63 | 1·59 | 1·57 | 1·53 | 1·48 | 1·42 | 1·33 |
| 25 | 1·73 | 1·72 | 1·66 | 1·62 | 1·59 | 1·56 | 1·52 | 1·47 | 1·41 | 1·32 |
| 26 | 1·73 | 1·71 | 1·66 | 1·62 | 1·58 | 1·56 | 1·52 | 1·47 | 1·40 | 1·31 |
| 27 | 1·73 | 1·71 | 1·66 | 1·61 | 1·58 | 1·55 | 1·51 | 1·46 | 1·40 | 1·30 |
| 28 | 1·72 | 1·71 | 1·65 | 1·61 | 1·57 | 1·55 | 1·51 | 1·46 | 1·39 | 1·30 |
| 29 | 1·72 | 1·70 | 1·65 | 1·60 | 1·57 | 1·54 | 1·50 | 1·45 | 1·39 | 1·29 |
| 30 | 1·72 | 1·70 | 1·64 | 1·60 | 1·57 | 1·54 | 1·50 | 1·45 | 1·38 | 1·28 |
| 40 | 1·70 | 1·68 | 1·62 | 1·57 | 1·54 | 1·51 | 1·47 | 1·41 | 1·34 | 1·24 |
| 60 | 1·68 | 1·65 | 1·59 | 1·55 | 1·51 | 1·48 | 1·44 | 1·38 | 1·31 | 1·18 |
| 120 | 1·66 | 1·63 | 1·57 | 1·52 | 1·48 | 1·45 | 1·41 | 1·35 | 1·27 | 1·12 |
| ∞ | 1·64 | 1·61 | 1·55 | 1·50 | 1·46 | 1·43 | 1·38 | 1·32 | 1·23 | 1·00 |

Lower 20 per cent. points are found by interchange of $n_1$ and $n_2$, *i.e.* $n_1$ must always correspond with the greater mean square.

## TABLE V. DISTRIBUTION OF *z*—*continued*

### 10 Per Cent. Points

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1·8427 | 1·9510 | 1·9907 | 2·0112 | 2·0236 | 2·0320 | 2·0425 | 2·0530 | 2·0636 | 2·0742 |
| 2 | 1·0716 | 1·0986 | 1·1075 | 1·1120 | 1·1146 | 1·1164 | 1·1186 | 1·1208 | 1·1230 | 1·1252 |
| 3 | ·8558 | ·8489 | ·8423 | ·8379 | ·8347 | ·8324 | ·8293 | ·8258 | ·8221 | ·8179 |
| 4 | ·7570 | ·7322 | ·7164 | ·7064 | ·6994 | ·6944 | ·6875 | ·6799 | ·6716 | ·6623 |
| 5 | ·7006 | ·6648 | ·6432 | ·6293 | ·6196 | ·6125 | ·6029 | ·5921 | ·5801 | ·5665 |
| 6 | ·6643 | ·6211 | ·5953 | ·5786 | ·5669 | ·5583 | ·5465 | ·5332 | ·5181 | ·5007 |
| 7 | ·6390 | ·5905 | ·5615 | ·5427 | ·5295 | ·5197 | ·5061 | ·4907 | ·4730 | ·4523 |
| 8 | ·6203 | ·5678 | ·5364 | ·5160 | ·5015 | ·4907 | ·4757 | ·4585 | ·4386 | ·4148 |
| 9 | ·6060 | ·5504 | ·5171 | ·4953 | ·4798 | ·4682 | ·4520 | ·4333 | ·4114 | ·3849 |
| 10 | ·5947 | ·5366 | ·5017 | ·4788 | ·4624 | ·4502 | ·4330 | ·4130 | ·3893 | ·3602 |
| 11 | ·5855 | ·5253 | ·4892 | ·4653 | ·4483 | ·4355 | ·4173 | ·3962 | ·3710 | ·3395 |
| 12 | ·5779 | ·5160 | ·4788 | ·4541 | ·4365 | ·4231 | ·4043 | ·3821 | ·3555 | ·3219 |
| 13 | ·5715 | ·5082 | ·4701 | ·4447 | ·4265 | ·4127 | ·3932 | ·3702 | ·3422 | ·3066 |
| 14 | ·5661 | ·5015 | ·4626 | ·4366 | ·4180 | ·4038 | ·3836 | ·3598 | ·3308 | ·2931 |
| 15 | ·5614 | ·4957 | ·4561 | ·4296 | ·4106 | ·3961 | ·3754 | ·3508 | ·3207 | ·2813 |
| 16 | ·5572 | ·4907 | ·4504 | ·4235 | ·4041 | ·3893 | ·3681 | ·3429 | ·3118 | ·2706 |
| 17 | ·5537 | ·4863 | ·4455 | ·4181 | ·3984 | ·3833 | ·3617 | ·3359 | ·3038 | ·2611 |
| 18 | ·5505 | ·4823 | ·4411 | ·4134 | ·3933 | ·3780 | ·3560 | ·3296 | ·2967 | ·2524 |
| 19 | ·5476 | ·4788 | ·4371 | ·4091 | ·3887 | ·3732 | ·3508 | ·3240 | ·2904 | ·2445 |
| 20 | ·5451 | ·4757 | ·4336 | ·4052 | ·3846 | ·3689 | ·3462 | ·3189 | ·2846 | ·2373 |
| 21 | ·5427 | ·4728 | ·4304 | ·4017 | ·3809 | ·3650 | ·3420 | ·3143 | ·2793 | ·2307 |
| 22 | ·5407 | ·4703 | ·4275 | ·3986 | ·3776 | ·3615 | ·3382 | ·3101 | ·2744 | ·2245 |
| 23 | ·5388 | ·4679 | ·4248 | ·3957 | ·3745 | ·3582 | ·3347 | ·3062 | ·2700 | ·2188 |
| 24 | ·5370 | ·4657 | ·4224 | ·3931 | ·3717 | ·3553 | ·3315 | ·3027 | ·2659 | ·2135 |
| 25 | ·5354 | ·4638 | ·4201 | ·3906 | ·3691 | ·3526 | ·3286 | ·2994 | ·2621 | ·2086 |
| 26 | ·5339 | ·4620 | ·4181 | ·3884 | ·3667 | ·3500 | ·3258 | ·2964 | ·2585 | ·2039 |
| 27 | ·5326 | ·4603 | ·4162 | ·3863 | ·3645 | ·3477 | ·3233 | ·2936 | ·2553 | ·1996 |
| 28 | ·5313 | ·4587 | ·4144 | ·3844 | ·3624 | ·3455 | ·3210 | ·2910 | ·2522 | ·1955 |
| 29 | ·5301 | ·4572 | ·4128 | ·3826 | ·3605 | ·3435 | ·3188 | ·2885 | ·2493 | ·1916 |
| 30 | ·5290 | ·4559 | ·4112 | ·3809 | ·3587 | ·3416 | ·3167 | ·2863 | ·2466 | ·1880 |
| 40 | ·5211 | ·4461 | ·4001 | ·3688 | ·3458 | ·3280 | ·3019 | ·2696 | ·2268 | ·1599 |
| 60 | ·5133 | ·4363 | ·3891 | ·3567 | ·3328 | ·3142 | ·2868 | ·2526 | ·2063 | ·1279 |
| 120 | ·5054 | ·4266 | ·3781 | ·3446 | ·3198 | ·3005 | ·2717 | ·2354 | ·1848 | ·0881 |
| ∞ | ·4976 | ·4170 | ·3671 | ·3326 | ·3069 | ·2866 | ·2565 | ·2178 | ·1622 | 0 |

For high values of $n_1$ and $n_2$, $z$ (10 per cent.) $= \dfrac{1·2816}{\sqrt{h-0·8}} - 0·6071\left(\dfrac{1}{n_1} - \dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h} = \dfrac{1}{n_1} + \dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of $n_1$ and $n_2$ are taken.

## TABLE V. VARIANCE RATIO—*continued*

### 10 Per Cent. Points of $e^{2z}$

| $n_1$ / $n_2$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 39·86 | 49·50 | 53·59 | 55·83 | 57·24 | 58·20 | 59·44 | 60·70 | 62·00 | 63·33 |
| 2 | 8·53 | 9·00 | 9·16 | 9·24 | 9·29 | 9·33 | 9·37 | 9·41 | 9·45 | 9·49 |
| 3 | 5·54 | 5·46 | 5·39 | 5·34 | 5·31 | 5·28 | 5·25 | 5·22 | 5·18 | 5·13 |
| 4 | 4·54 | 4·32 | 4·19 | 4·11 | 4·05 | 4·01 | 3·95 | 3·90 | 3·83 | 3·76 |
| 5 | 4·06 | 3·78 | 3·62 | 3·52 | 3·45 | 3·40 | 3·34 | 3·27 | 3·19 | 3·10 |
| 6 | 3·78 | 3·46 | 3·29 | 3·18 | 3·11 | 3·05 | 2·98 | 2·90 | 2·82 | 2·72 |
| 7 | 3·59 | 3·26 | 3·07 | 2·96 | 2·88 | 2·83 | 2·75 | 2·67 | 2·58 | 2·47 |
| 8 | 3·46 | 3·11 | 2·92 | 2·81 | 2·73 | 2·67 | 2·59 | 2·50 | 2·40 | 2·29 |
| 9 | 3·36 | 3·01 | 2·81 | 2·69 | 2·61 | 2·55 | 2·47 | 2·38 | 2·28 | 2·16 |
| 10 | 3·28 | 2·92 | 2·73 | 2·61 | 2·52 | 2·46 | 2·38 | 2·28 | 2·18 | 2·06 |
| 11 | 3·23 | 2·86 | 2·66 | 2·54 | 2·45 | 2·39 | 2·30 | 2·21 | 2·10 | 1·97 |
| 12 | 3·18 | 2·81 | 2·61 | 2·48 | 2·39 | 2·33 | 2·24 | 2·15 | 2·04 | 1·90 |
| 13 | 3·14 | 2·76 | 2·56 | 2·43 | 2·35 | 2·28 | 2·20 | 2·10 | 1·98 | 1·85 |
| 14 | 3·10 | 2·73 | 2·52 | 2·39 | 2·31 | 2·24 | 2·15 | 2·05 | 1·94 | 1·80 |
| 15 | 3·07 | 2·70 | 2·49 | 2·36 | 2·27 | 2·21 | 2·12 | 2·02 | 1·90 | 1·76 |
| 16 | 3·05 | 2·67 | 2·46 | 2·33 | 2·24 | 2·18 | 2·09 | 1·99 | 1·87 | 1·72 |
| 17 | 3·03 | 2·64 | 2·44 | 2·31 | 2·22 | 2·15 | 2·06 | 1·96 | 1·84 | 1·69 |
| 18 | 3·01 | 2·62 | 2·42 | 2·29 | 2·20 | 2·13 | 2·04 | 1·93 | 1·81 | 1·66 |
| 19 | 2·99 | 2·61 | 2·40 | 2·27 | 2·18 | 2·11 | 2·02 | 1·91 | 1·79 | 1·63 |
| 20 | 2·97 | 2·59 | 2·38 | 2·25 | 2·16 | 2·09 | 2·00 | 1·89 | 1·77 | 1·61 |
| 21 | 2·96 | 2·57 | 2·36 | 2·23 | 2·14 | 2·08 | 1·98 | 1·88 | 1·75 | 1·59 |
| 22 | 2·95 | 2·56 | 2·35 | 2·22 | 2·13 | 2·06 | 1·97 | 1·86 | 1·73 | 1·57 |
| 23 | 2·94 | 2·55 | 2·34 | 2·21 | 2·11 | 2·05 | 1·95 | 1·84 | 1·72 | 1·55 |
| 24 | 2·93 | 2·54 | 2·33 | 2·19 | 2·10 | 2·04 | 1·94 | 1·83 | 1·70 | 1·53 |
| 25 | 2·92 | 2·53 | 2·32 | 2·18 | 2·09 | 2·02 | 1·93 | 1·82 | 1·69 | 1·52 |
| 26 | 2·91 | 2·52 | 2·31 | 2·17 | 2·08 | 2·01 | 1·92 | 1·81 | 1·68 | 1·50 |
| 27 | 2·90 | 2·51 | 2·30 | 2·17 | 2·07 | 2·00 | 1·91 | 1·80 | 1·67 | 1·49 |
| 28 | 2·89 | 2·50 | 2·29 | 2·16 | 2·06 | 2·00 | 1·90 | 1·79 | 1·66 | 1·48 |
| 29 | 2·89 | 2·50 | 2·28 | 2·15 | 2·06 | 1·99 | 1·89 | 1·78 | 1·65 | 1·47 |
| 30 | 2·88 | 2·49 | 2·28 | 2·14 | 2·05 | 1·98 | 1·88 | 1·77 | 1·64 | 1·46 |
| 40 | 2·84 | 2·44 | 2·23 | 2·09 | 2·00 | 1·93 | 1·83 | 1·71 | 1·57 | 1·38 |
| 60 | 2·79 | 2·39 | 2·18 | 2·04 | 1·95 | 1·87 | 1·77 | 1·66 | 1·51 | 1·29 |
| 120 | 2·75 | 2·35 | 2·13 | 1·99 | 1·90 | 1·82 | 1·72 | 1·60 | 1·45 | 1·19 |
| ∞ | 2·71 | 2·30 | 2·08 | 1·94 | 1·85 | 1·77 | 1·67 | 1·55 | 1·38 | 1·00 |

Lower 10 per cent. points are found by interchange of $n_1$ and $n_2$, *i.e.* $n_1$ must always correspond with the greater mean square.

## TABLE V. DISTRIBUTION OF $z$—*continued*

### 5 Per Cent. Points

| $n_2 \backslash n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2·5421 | 2·6479 | 2·6870 | 2·7071 | 2·7194 | 2·7276 | 2·7380 | 2·7484 | 2·7588 | 2·7693 |
| 2 | 1·4592 | 1·4722 | 1·4765 | 1·4787 | 1·4800 | 1·4808 | 1·4819 | 1·4830 | 1·4840 | 1·4851 |
| 3 | 1·1577 | 1·1284 | 1·1137 | 1·1051 | 1·0994 | 1·0953 | 1·0899 | 1·0842 | 1·0781 | 1·0716 |
| 4 | 1·0212 | ·9690 | ·9429 | ·9272 | ·9168 | ·9093 | ·8993 | ·8885 | ·8767 | ·8639 |
| 5 | ·9441 | ·8777 | ·8441 | ·8236 | ·8097 | ·7997 | ·7862 | ·7714 | ·7550 | ·7368 |
| 6 | ·8948 | ·8188 | ·7798 | ·7558 | ·7394 | ·7274 | ·7112 | ·6931 | ·6729 | ·6499 |
| 7 | ·8606 | ·7777 | ·7347 | ·7080 | ·6896 | ·6761 | ·6576 | ·6369 | ·6134 | ·5862 |
| 8 | ·8355 | ·7475 | ·7014 | ·6725 | ·6525 | ·6378 | ·6175 | ·5945 | ·5682 | ·5371 |
| 9 | ·8163 | ·7242 | ·6757 | ·6450 | ·6238 | ·6080 | ·5862 | ·5613 | ·5324 | ·4979 |
| 10 | ·8012 | ·7058 | ·6553 | ·6232 | ·6009 | ·5843 | ·5611 | ·5346 | ·5035 | ·4657 |
| 11 | ·7889 | ·6909 | ·6387 | ·6055 | ·5822 | ·5648 | ·5406 | ·5126 | ·4795 | ·4387 |
| 12 | ·7788 | ·6786 | ·6250 | ·5907 | ·5666 | ·5487 | ·5234 | ·4941 | ·4592 | ·4156 |
| 13 | ·7703 | ·6682 | ·6134 | ·5783 | ·5535 | ·5350 | ·5089 | ·4785 | ·4419 | ·3957 |
| 14 | ·7630 | ·6594 | ·6036 | ·5677 | ·5423 | ·5233 | ·4964 | ·4649 | ·4269 | ·3782 |
| 15 | ·7568 | ·6518 | ·5950 | ·5585 | ·5326 | ·5131 | ·4855 | ·4532 | ·4138 | ·3628 |
| 16 | ·7514 | ·6451 | ·5876 | ·5505 | ·5241 | ·5042 | ·4760 | ·4428 | ·4022 | ·3490 |
| 17 | ·7466 | ·6393 | ·5811 | ·5434 | ·5166 | ·4964 | ·4676 | ·4337 | ·3919 | ·3366 |
| 18 | ·7424 | ·6341 | ·5753 | ·5371 | ·5099 | ·4894 | ·4602 | ·4255 | ·3827 | ·3253 |
| 19 | ·7386 | ·6295 | ·5701 | ·5315 | ·5040 | ·4832 | ·4535 | ·4182 | ·3743 | ·3151 |
| 20 | ·7352 | ·6254 | ·5654 | ·5265 | ·4986 | ·4776 | ·4474 | ·4116 | ·3668 | ·3057 |
| 21 | ·7322 | ·6216 | ·5612 | ·5219 | ·4938 | ·4725 | ·4420 | ·4055 | ·3599 | ·2971 |
| 22 | ·7294 | ·6182 | ·5574 | ·5178 | ·4894 | ·4679 | ·4370 | ·4001 | ·3536 | ·2892 |
| 23 | ·7269 | ·6151 | ·5540 | ·5140 | ·4854 | ·4636 | ·4325 | ·3950 | ·3478 | ·2818 |
| 24 | ·7246 | ·6123 | ·5508 | ·5106 | ·4817 | ·4598 | ·4283 | ·3904 | ·3425 | ·2749 |
| 25 | ·7225 | ·6097 | ·5478 | ·5074 | ·4783 | ·4562 | ·4244 | ·3862 | ·3376 | ·2685 |
| 26 | ·7205 | ·6073 | ·5451 | ·5045 | ·4752 | ·4529 | ·4209 | ·3823 | ·3330 | ·2625 |
| 27 | ·7187 | ·6051 | ·5427 | ·5017 | ·4723 | ·4499 | ·4176 | ·3786 | ·3287 | ·2569 |
| 28 | ·7171 | ·6030 | ·5403 | ·4992 | ·4696 | ·4471 | ·4146 | ·3752 | ·3248 | ·2516 |
| 29 | ·7155 | ·6011 | ·5382 | ·4969 | ·4671 | ·4444 | ·4117 | ·3720 | ·3211 | ·2466 |
| 30 | ·7141 | ·5994 | ·5362 | ·4947 | ·4648 | ·4420 | ·4090 | ·3691 | ·3176 | ·2419 |
| 40 | ·7037 | ·5866 | ·5217 | ·4789 | ·4479 | ·4242 | ·3897 | ·3475 | ·2920 | ·2057 |
| 60 | ·6933 | ·5738 | ·5073 | ·4632 | ·4311 | ·4064 | ·3702 | ·3255 | ·2654 | ·1644 |
| 120 | ·6830 | ·5611 | ·4930 | ·4475 | ·4143 | ·3885 | ·3506 | ·3032 | ·2376 | ·1131 |
| ∞ | ·6729 | ·5486 | ·4787 | ·4319 | ·3974 | ·37[illegible]6 | ·3309 | ·2804 | ·2085 | 0 |

For high values of $n_1$ and $n_2$, $z$ (5 per cent.) $= \dfrac{1{\cdot}6449}{\sqrt{h-1}} - 0{\cdot}7843 \left(\dfrac{1}{n_1} - \dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h} = \dfrac{1}{n_1} + \dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of $n_1$ and $n_2$ are taken.

## TABLE V. VARIANCE RATIO—*continued*

### 5 Per Cent. Points of $e^{2z}$

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 161·4 | 199·5 | 215·7 | 224·6 | 230·2 | 234·0 | 238·9 | 243·9 | 249·0 | 254·3 |
| 2 | 18·51 | 19·00 | 19·16 | 19·25 | 19·30 | 19·33 | 19·37 | 19·41 | 19·45 | 19·50 |
| 3 | 10·13 | 9·55 | 9·28 | 9·12 | 9·01 | 8·94 | 8·84 | 8·74 | 8·64 | 8·53 |
| 4 | 7·71 | 6·94 | 6·59 | 6·39 | 6·26 | 6·16 | 6·04 | 5·91 | 5·77 | 5·63 |
| 5 | 6·61 | 5·79 | 5·41 | 5·19 | 5·05 | 4·95 | 4·82 | 4·68 | 4·53 | 4·36 |
| 6 | 5·99 | 5·14 | 4·76 | 4·53 | 4·39 | 4·28 | 4·15 | 4·00 | 3·84 | 3·67 |
| 7 | 5·59 | 4·74 | 4·35 | 4·12 | 3·97 | 3·87 | 3·73 | 3·57 | 3·41 | 3·23 |
| 8 | 5·32 | 4·46 | 4·07 | 3·84 | 3·69 | 3·58 | 3·44 | 3·28 | 3·12 | 2·93 |
| 9 | 5·12 | 4·26 | 3·86 | 3·63 | 3·48 | 3·37 | 3·23 | 3·07 | 2·90 | 2·71 |
| 10 | 4·96 | 4·10 | 3·71 | 3·48 | 3·33 | 3·22 | 3·07 | 2·91 | 2·74 | 2·54 |
| 11 | 4·84 | 3·98 | 3·59 | 3·36 | 3·20 | 3·09 | 2·95 | 2·79 | 2·61 | 2·40 |
| 12 | 4·75 | 3·88 | 3·49 | 3·26 | 3·11 | 3·00 | 2·85 | 2·69 | 2·50 | 2·30 |
| 13 | 4·67 | 3·80 | 3·41 | 3·18 | 3·02 | 2·92 | 2·77 | 2·60 | 2·42 | 2·21 |
| 14 | 4·60 | 3·74 | 3·34 | 3·11 | 2·96 | 2·85 | 2·70 | 2·53 | 2·35 | 2·13 |
| 15 | 4·54 | 3·68 | 3·29 | 3·06 | 2·90 | 2·79 | 2·64 | 2·48 | 2·29 | 2·07 |
| 16 | 4·49 | 3·63 | 3·24 | 3·01 | 2·85 | 2·74 | 2·59 | 2·42 | 2·24 | 2·01 |
| 17 | 4·45 | 3·59 | 3·20 | 2·96 | 2·81 | 2·70 | 2·55 | 2·38 | 2·19 | 1·96 |
| 18 | 4·41 | 3·55 | 3·16 | 2·93 | 2·77 | 2·66 | 2·51 | 2·34 | 2·15 | 1·92 |
| 19 | 4·38 | 3·52 | 3·13 | 2·90 | 2·74 | 2·63 | 2·48 | 2·31 | 2·11 | 1·88 |
| 20 | 4·35 | 3·49 | 3·10 | 2·87 | 2·71 | 2·60 | 2·45 | 2·28 | 2·08 | 1·84 |
| 21 | 4·32 | 3·47 | 3·07 | 2·84 | 2·68 | 2·57 | 2·42 | 2·25 | 2·05 | 1·81 |
| 22 | 4·30 | 3·44 | 3·05 | 2·82 | 2·66 | 2·55 | 2·40 | 2·23 | 2·03 | 1·78 |
| 23 | 4·28 | 3·42 | 3·03 | 2·80 | 2·64 | 2·53 | 2·38 | 2·20 | 2·00 | 1·76 |
| 24 | 4·26 | 3·40 | 3·01 | 2·78 | 2·62 | 2·51 | 2·36 | 2·18 | 1·98 | 1·73 |
| 25 | 4·24 | 3·38 | 2·99 | 2·76 | 2·60 | 2·49 | 2·34 | 2·16 | 1·96 | 1·71 |
| 26 | 4·22 | 3·37 | 2·98 | 2·74 | 2·59 | 2·47 | 2·32 | 2·15 | 1·95 | 1·69 |
| 27 | 4·21 | 3·35 | 2·96 | 2·73 | 2·57 | 2·46 | 2·30 | 2·13 | 1·93 | 1·67 |
| 28 | 4·20 | 3·34 | 2·95 | 2·71 | 2·56 | 2·44 | 2·29 | 2·12 | 1·91 | 1·65 |
| 29 | 4·18 | 3·33 | 2·93 | 2·70 | 2·54 | 2·43 | 2·28 | 2·10 | 1·90 | 1·64 |
| 30 | 4·17 | 3·32 | 2·92 | 2·69 | 2·53 | 2·42 | 2·27 | 2·09 | 1·89 | 1·62 |
| 40 | 4·08 | 3·23 | 2·84 | 2·61 | 2·45 | 2·34 | 2·18 | 2·00 | 1·79 | 1·51 |
| 60 | 4·00 | 3·15 | 2·76 | 2·52 | 2·37 | 2·25 | 2·10 | 1·92 | 1·70 | 1·39 |
| 120 | 3·92 | 3·07 | 2·68 | 2·45 | 2·29 | 2·17 | 2·02 | 1·83 | 1·61 | 1·25 |
| ∞ | 3·84 | 2·99 | 2·60 | 2·37 | 2·21 | 2·10 | 1·94 | 1·75 | 1·52 | 1·00 |

Lower 5 per cent. points are found by interchange of $n_1$ and $n_2$, *i.e.* $n_1$ must always correspond with the greater mean square.

## TABLE V. DISTRIBUTION OF *z*—*continued*

### 1 Per Cent. Points

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4·1535 | 4·2585 | 4·2974 | 4·3175 | 4·3297 | 4·3379 | 4·3482 | 4·3585 | 4·3689 | 4·3794 |
| 2 | 2·2950 | 2·2976 | 2·2984 | 2·2988 | 2·2991 | 2·2992 | 2·2994 | 2·2997 | 2·2999 | 2·3001 |
| 3 | 1·7649 | 1·7140 | 1·6915 | 1·6786 | 1·6703 | 1·6645 | 1·6569 | 1·6489 | 1·6404 | 1·6314 |
| 4 | 1·5270 | 1·4452 | 1·4075 | 1·3856 | 1·3711 | 1·3609 | 1·3473 | 1·3327 | 1·3170 | 1·3000 |
| 5 | 1·3943 | 1·2929 | 1·2449 | 1·2164 | 1·1974 | 1·1838 | 1·1656 | 1·1457 | 1·1239 | 1·0997 |
| 6 | 1·3103 | 1·1955 | 1·1401 | 1·1068 | 1·0843 | 1·0680 | 1·0460 | 1·0218 | ·9948 | ·9643 |
| 7 | 1·2526 | 1·1281 | 1·0672 | 1·0300 | 1·0048 | ·9864 | ·9614 | ·9335 | ·9020 | ·8658 |
| 8 | 1·2106 | 1·0787 | 1·0135 | ·9734 | ·9459 | ·9259 | ·8983 | ·8673 | ·8319 | ·7904 |
| 9 | 1·1786 | 1·0411 | ·9724 | ·9299 | ·9006 | ·8791 | ·8494 | ·8157 | ·7769 | ·7305 |
| 10 | 1·1535 | 1·0114 | ·9399 | ·8954 | ·8646 | ·8419 | ·8104 | ·7744 | ·7324 | ·6816 |
| 11 | 1·1333 | ·9874 | ·9136 | ·8674 | ·8354 | ·8116 | ·7785 | ·7405 | ·6958 | ·6408 |
| 12 | 1·1166 | ·9677 | ·8919 | ·8443 | ·8111 | ·7864 | ·7520 | ·7122 | ·6649 | ·6061 |
| 13 | 1·1027 | ·9511 | ·8737 | ·8248 | ·7907 | ·7652 | ·7295 | ·6882 | ·6386 | ·5761 |
| 14 | 1·0909 | ·9370 | ·8581 | ·8082 | ·7732 | ·7471 | ·7103 | ·6675 | ·6159 | ·5500 |
| 15 | 1·0807 | ·9249 | ·8448 | ·7939 | ·7582 | ·7314 | ·6937 | ·6496 | ·5961 | ·5269 |
| 16 | 1·0719 | ·9144 | ·8331 | ·7814 | ·7450 | ·7177 | ·6791 | ·6339 | ·5786 | ·5064 |
| 17 | 1·0641 | ·9051 | ·8229 | ·7705 | ·7335 | ·7057 | ·6663 | ·6199 | ·5630 | ·4879 |
| 18 | 1·0572 | ·8970 | ·8138 | ·7607 | ·7232 | ·6950 | ·6549 | ·6075 | ·5491 | ·4712 |
| 19 | 1·0511 | ·8897 | ·8057 | ·7521 | ·7140 | ·6854 | ·6447 | ·5964 | ·5366 | ·4560 |
| 20 | 1·0457 | ·8831 | ·7985 | ·7443 | ·7058 | ·6768 | ·6355 | ·5864 | ·5253 | ·4421 |
| 21 | 1·0408 | ·8772 | ·7920 | ·7372 | ·6984 | ·6690 | ·6272 | ·5773 | ·5150 | ·4294 |
| 22 | 1·0363 | ·8719 | ·7860 | ·7309 | ·6916 | ·6620 | ·6196 | ·5691 | ·5056 | ·4176 |
| 23 | 1·0322 | ·8670 | ·7806 | ·7251 | ·6855 | ·6555 | ·6127 | ·5615 | ·4969 | ·4068 |
| 24 | 1·0285 | ·8626 | ·7757 | ·7197 | ·6799 | ·6496 | ·6064 | ·5545 | ·4890 | ·3967 |
| 25 | 1·0251 | ·8585 | ·7712 | ·7148 | ·6747 | ·6442 | ·6006 | ·5481 | ·4816 | ·3872 |
| 26 | 1·0220 | ·8548 | ·7670 | ·7103 | ·6699 | ·6392 | ·5952 | ·5422 | ·4748 | ·3784 |
| 27 | 1·0191 | ·8513 | ·7631 | ·7062 | ·6655 | ·6346 | ·5902 | ·5367 | ·4685 | ·3701 |
| 28 | 1·0164 | ·8481 | ·7595 | ·7023 | ·6614 | ·6303 | ·5856 | ·5316 | ·4626 | ·3624 |
| 29 | 1·0139 | ·8451 | ·7562 | ·6987 | ·6576 | ·6263 | ·5813 | ·5269 | ·4570 | ·3550 |
| 30 | 1·0116 | ·8423 | ·7531 | ·6954 | ·6540 | ·6226 | ·5773 | ·5224 | ·4519 | ·3481 |
| 40 | ·9949 | ·8223 | ·7307 | ·6712 | ·6283 | ·5956 | ·5481 | ·4901 | ·4138 | ·2952 |
| 60 | ·9784 | ·8025 | ·7086 | ·6472 | ·6028 | ·5687 | ·5189 | ·4574 | ·3746 | ·2352 |
| 120 | ·9622 | ·7829 | ·6867 | ·6234 | ·5774 | ·5419 | ·4897 | ·4243 | ·3339 | ·1612 |
| ∞ | ·9462 | ·7636 | ·6651 | ·5999 | ·5522 | ·5152 | ·4604 | ·3908 | ·2913 | 0 |

For high values of $n_1$ and $n_2$, $z$ (1 per cent.) $= \dfrac{2{\cdot}3263}{\sqrt{h-1{\cdot}4}} - 1{\cdot}235\left(\dfrac{1}{n_1} - \dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h} = \dfrac{1}{n_1} + \dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of $n_1$ and $n_2$ are taken.

## TABLE V. VARIANCE RATIO—*continued*

### 1 Per Cent. Points of $e^{2z}$

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4052 | 4999 | 5403 | 5625 | 5764 | 5859 | 5982 | 6106 | 6234 | 6366 |
| 2 | 98·50 | 99·00 | 99·17 | 99·25 | 99·30 | 99·33 | 99·37 | 99·42 | 99·46 | 99·50 |
| 3 | 34·12 | 30·82 | 29·46 | 28·71 | 28·24 | 27·91 | 27·49 | 27·05 | 26·60 | 26·12 |
| 4 | 21·20 | 18·00 | 16·69 | 15·98 | 15·52 | 15·21 | 14·80 | 14·37 | 13·93 | 13·46 |
| 5 | 16·26 | 13·27 | 12·06 | 11·39 | 10·97 | 10·67 | 10·29 | 9·89 | 9·47 | 9·02 |
| 6 | 13·74 | 10·92 | 9·78 | 9·15 | 8·75 | 8·47 | 8·10 | 7·72 | 7·31 | 6·88 |
| 7 | 12·25 | 9·55 | 8·45 | 7·85 | 7·46 | 7·19 | 6·84 | 6·47 | 6·07 | 5·65 |
| 8 | 11·26 | 8·65 | 7·59 | 7·01 | 6·63 | 6·37 | 6·03 | 5·67 | 5·28 | 4·86 |
| 9 | 10·56 | 8·02 | 6·99 | 6·42 | 6·06 | 5·80 | 5·47 | 5·11 | 4·73 | 4·31 |
| 10 | 10·04 | 7·56 | 6·55 | 5·99 | 5·64 | 5·39 | 5·06 | 4·71 | 4·33 | 3·91 |
| 11 | 9·65 | 7·20 | 6·22 | 5·67 | 5·32 | 5·07 | 4·74 | 4·40 | 4·02 | 3·60 |
| 12 | 9·33 | 6·93 | 5·95 | 5·41 | 5·06 | 4·82 | 4·50 | 4·16 | 3·78 | 3·36 |
| 13 | 9·07 | 6·70 | 5·74 | 5·20 | 4·86 | 4·62 | 4·30 | 3·96 | 3·59 | 3·16 |
| 14 | 8·86 | 6·51 | 5·56 | 5·03 | 4·69 | 4·46 | 4·14 | 3·80 | 3·43 | 3·00 |
| 15 | 8·68 | 6·36 | 5·42 | 4·89 | 4·56 | 4·32 | 4·00 | 3·67 | 3·29 | 2·87 |
| 16 | 8·53 | 6·23 | 5·29 | 4·77 | 4·44 | 4·20 | 3·89 | 3·55 | 3·18 | 2·75 |
| 17 | 8·40 | 6·11 | 5·18 | 4·67 | 4·34 | 4·10 | 3·79 | 3·45 | 3·08 | 2·65 |
| 18 | 8·28 | 6·01 | 5·09 | 4·58 | 4·25 | 4·01 | 3·71 | 3·37 | 3·00 | 2·57 |
| 19 | 8·18 | 5·93 | 5·01 | 4·50 | 4·17 | 3·94 | 3·63 | 3·30 | 2·92 | 2·49 |
| 20 | 8·10 | 5·85 | 4·94 | 4·43 | 4·10 | 3·87 | 3·56 | 3·23 | 2·86 | 2·42 |
| 21 | 8·02 | 5·78 | 4·87 | 4·37 | 4·04 | 3·81 | 3·51 | 3·17 | 2·80 | 2·36 |
| 22 | 7·94 | 5·72 | 4·82 | 4·31 | 3·99 | 3·76 | 3·45 | 3·12 | 2·75 | 2·31 |
| 23 | 7·88 | 5·66 | 4·76 | 4·26 | 3·94 | 3·71 | 3·41 | 3·07 | 2·70 | 2·26 |
| 24 | 7·82 | 5·61 | 4·72 | 4·22 | 3·90 | 3·67 | 3·36 | 3·03 | 2·66 | 2·21 |
| 25 | 7·77 | 5·57 | 4·68 | 4·18 | 3·86 | 3·63 | 3·32 | 2·99 | 2·62 | 2·17 |
| 26 | 7·72 | 5·53 | 4·64 | 4·14 | 3·82 | 3·59 | 3·29 | 2·96 | 2·58 | 2·13 |
| 27 | 7·68 | 5·49 | 4·60 | 4·11 | 3·78 | 3·56 | 3·26 | 2·93 | 2·55 | 2·10 |
| 28 | 7·64 | 5·45 | 4·57 | 4·07 | 3·75 | 3·53 | 3·23 | 2·90 | 2·52 | 2·06 |
| 29 | 7·60 | 5·42 | 4·54 | 4·04 | 3·73 | 3·50 | 3·20 | 2·87 | 2·49 | 2·03 |
| 30 | 7·56 | 5·39 | 4·51 | 4·02 | 3·70 | 3·47 | 3·17 | 2·84 | 2·47 | 2·01 |
| 40 | 7·31 | 5·18 | 4·31 | 3·83 | 3·51 | 3·29 | 2·99 | 2·66 | 2·29 | 1·80 |
| 60 | 7·08 | 4·98 | 4·13 | 3·65 | 3·34 | 3·12 | 2·82 | 2·50 | 2·12 | 1·60 |
| 120 | 6·85 | 4·79 | 3·95 | 3·48 | 3·17 | 2·96 | 2·66 | 2·34 | 1·95 | 1·38 |
| ∞ | 6·64 | 4·60 | 3·78 | 3·32 | 3·02 | 2·80 | 2·51 | 2·18 | 1·79 | 1·00 |

Lower 1 per cent. points are found by interchange of $n_1$ and $n_2$, *i.e.* $n_1$ must always correspond with the greater mean square.

## TABLE V. DISTRIBUTION OF *z*—*continued*

### 0·1 Per Cent. Points (Colcord and Deming)

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6·4562 | 6·5612 | 6·6000 | 6·6201 | 6·6323 | 6·6405 | 6·6508 | 6·6611 | 6·6715 | 6·6819 |
| 2 | 3·4531 | 3·4534 | 3·4535 | 3·4535 | 3·4535 | 3·4535 | 3·4536 | 3·4536 | 3·4536 | 3·4536 |
| 3 | 2·5591 | 2·5003 | 2·4748 | 2·4603 | 2·4511 | 2·4446 | 2·4361 | 2·4272 | 2·4179 | 2·4080 |
| 4 | 2·1529 | 2·0574 | 2·0143 | 1·9892 | 1·9728 | 1·9612 | 1·9459 | 1·9294 | 1·9118 | 1·8927 |
| 5 | 1·9270 | 1·8071 | 1·7513 | 1·7184 | 1·6964 | 1·6808 | 1·6598 | 1·6370 | 1·6121 | 1·5845 |
| 6 | 1·7849 | 1·6479 | 1·5828 | 1·5438 | 1·5175 | 1·4986 | 1·4730 | 1·4449 | 1·4136 | 1·3783 |
| 7 | 1·6879 | 1·5384 | 1·4662 | 1·4224 | 1·3927 | 1·3711 | 1·3417 | 1·3090 | 1·2721 | 1·2296 |
| 8 | 1·6177 | 1·4587 | 1·3809 | 1·3333 | 1·3008 | 1·2770 | 1·2443 | 1·2077 | 1·1659 | 1·1169 |
| 9 | 1·5646 | 1·3982 | 1·3160 | 1·2653 | 1·2304 | 1·2047 | 1·1694 | 1·1293 | 1·0830 | 1·0279 |
| 10 | 1·5232 | 1·3509 | 1·2650 | 1·2116 | 1·1748 | 1·1475 | 1·1098 | 1·0668 | 1·0165 | ·9557 |
| 11 | 1·4900 | 1·3128 | 1·2238 | 1·1683 | 1·1297 | 1·1012 | 1·0614 | 1·0157 | ·9619 | ·8957 |
| 12 | 1·4627 | 1·2814 | 1·1900 | 1·1326 | 1·0926 | 1·0628 | 1·0213 | ·9733 | ·9162 | ·8450 |
| 13 | 1·4400 | 1·2553 | 1·1616 | 1·1026 | 1·0614 | 1·0306 | ·9875 | ·9374 | ·8774 | ·8014 |
| 14 | 1·4208 | 1·2332 | 1·1376 | 1·0772 | 1·0348 | 1·0031 | ·9586 | ·9066 | ·8439 | ·7635 |
| 15 | 1·4043 | 1·2141 | 1·1169 | 1·0553 | 1·0119 | ·9795 | ·9336 | ·8800 | ·8147 | ·7301 |
| 16 | 1·3900 | 1·1976 | 1·0989 | 1·0362 | ·9920 | ·9588 | ·9119 | ·8567 | ·7891 | ·7005 |
| 17 | 1·3775 | 1·1832 | 1·0832 | 1·0195 | ·9745 | ·9407 | ·8927 | ·8361 | ·7664 | ·6740 |
| 18 | 1·3665 | 1·1704 | 1·0693 | 1·0047 | ·9590 | ·9246 | ·8757 | ·8178 | ·7462 | ·6502 |
| 19 | 1·3567 | 1·1591 | 1·0569 | ·9915 | ·9452 | ·9103 | ·8605 | ·8014 | ·7279 | ·6285 |
| 20 | 1·3480 | 1·1489 | 1·0458 | ·9798 | ·9329 | ·8974 | ·8469 | ·7867 | ·7115 | ·6086 |
| 21 | 1·3401 | 1·1398 | 1·0358 | ·9691 | ·9217 | ·8858 | ·8346 | ·7734 | ·6965 | ·5904 |
| 22 | 1·3329 | 1·1315 | 1·0268 | ·9595 | ·9116 | ·8753 | ·8234 | ·7612 | ·6828 | ·5738 |
| 23 | 1·3264 | 1·1240 | 1·0186 | ·9507 | ·9024 | ·8657 | ·8132 | ·7501 | ·6704 | ·5583 |
| 24 | 1·3205 | 1·1171 | 1·0111 | ·9427 | ·8939 | ·8569 | ·8038 | ·7400 | ·6589 | ·5440 |
| 25 | 1·3151 | 1·1108 | 1·0041 | ·9354 | ·8862 | ·8489 | ·7953 | ·7306 | ·6483 | ·5307 |
| 26 | 1·3101 | 1·1050 | ·9978 | ·9286 | ·8791 | ·8415 | ·7873 | ·7220 | ·6385 | ·5183 |
| 27 | 1·3055 | 1·0997 | ·9920 | ·9223 | ·8725 | ·8346 | ·7800 | ·7140 | ·6294 | ·5066 |
| 28 | 1·3013 | 1·0947 | ·9866 | ·9165 | ·8664 | ·8282 | ·7732 | ·7066 | ·6209 | ·4957 |
| 29 | 1·2973 | 1·0901 | ·9815 | ·9112 | ·8607 | ·8223 | ·7669 | ·6997 | ·6129 | ·4853 |
| 30 | 1·2936 | 1·0859 | ·9768 | ·9061 | ·8554 | ·8168 | ·7610 | ·6932 | ·6056 | ·4756 |
| 40 | 1·2672 | 1·0552 | ·9431 | ·8701 | ·8174 | ·7771 | ·7184 | ·6463 | ·5513 | ·4016 |
| 60 | 1·2413 | 1·0250 | ·9100 | ·8345 | ·7798 | ·7377 | ·6760 | ·5992 | ·4955 | ·3184 |
| 120 | 1·2159 | ·9954 | ·8773 | ·7994 | ·7426 | ·6986 | ·6338 | ·5519 | ·4381 | ·2170 |
| ∞ | 1·1910 | ·9663 | ·8453 | ·7648 | ·7059 | ·6599 | ·5917 | ·5044 | ·3786 | 0 |

For high values of $n_1$ and $n_2$, $z(0{\cdot}1 \text{ per cent.}) = \dfrac{3{\cdot}0902}{\sqrt{h-2{\cdot}1}} - 1{\cdot}925\left(\dfrac{1}{n_1} - \dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h} = \dfrac{1}{n_1} + \dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of $n_1$ and $n_2$ are taken.

## TABLE V. VARIANCE RATIO—*continued*

### 0·1 Per Cent. Points of $e^{2z}$

| $n_2$ \ $n_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 405284 | 500000 | 540379 | 562500 | 576405 | 585937 | 598144 | 610667 | 623497 | 636619 |
| 2 | 998·5 | 999·0 | 999·2 | 999·2 | 999·3 | 999·3 | 999·4 | 999·4 | 999·5 | 999·5 |
| 3 | 167·0 | 148·5 | 141·1 | 137·1 | 134·6 | 132·8 | 130·6 | 128·3 | 125·9 | 123·5 |
| 4 | 74·14 | 61·25 | 56·18 | 53·44 | 51·71 | 50·53 | 49·00 | 47·41 | 45·77 | 44·05 |
| 5 | 47·18 | 37·12 | 33·20 | 31·09 | 29·75 | 28·84 | 27·64 | 26·42 | 25·14 | 23·78 |
| 6 | 35·51 | 27·00 | 23·70 | 21·92 | 20·81 | 20·03 | 19·03 | 17·99 | 16·89 | 15·75 |
| 7 | 29·25 | 21·69 | 18·77 | 17·19 | 16·21 | 15·52 | 14·63 | 13·71 | 12·73 | 11·69 |
| 8 | 25·42 | 18·49 | 15·83 | 14·39 | 13·49 | 12·86 | 12·04 | 11·19 | 10·30 | 9·34 |
| 9 | 22·86 | 16·39 | 13·90 | 12·56 | 11·71 | 11·13 | 10·37 | 9·57 | 8·72 | 7·81 |
| 10 | 21·04 | 14·91 | 12·55 | 11·28 | 10·48 | 9·92 | 9·20 | 8·45 | 7·64 | 6·76 |
| 11 | 19·69 | 13·81 | 11·56 | 10·35 | 9·58 | 9·05 | 8·35 | 7·63 | 6·85 | 6·00 |
| 12 | 18·64 | 12·97 | 10·80 | 9·63 | 8·89 | 8·38 | 7·71 | 7·00 | 6·25 | 5·42 |
| 13 | 17·81 | 12·31 | 10·21 | 9·07 | 8·35 | 7·86 | 7·21 | 6·52 | 5·78 | 4·97 |
| 14 | 17·14 | 11·78 | 9·73 | 8·62 | 7·92 | 7·43 | 6·80 | 6·13 | 5·41 | 4·60 |
| 15 | 16·59 | 11·34 | 9·34 | 8·25 | 7·57 | 7·09 | 6·47 | 5·81 | 5·10 | 4·31 |
| 16 | 16·12 | 10·97 | 9·00 | 7·94 | 7·27 | 6·81 | 6·19 | 5·55 | 4·85 | 4·06 |
| 17 | 15·72 | 10·66 | 8·73 | 7·68 | 7·02 | 6·56 | 5·96 | 5·32 | 4·63 | 3·85 |
| 18 | 15·38 | 10·39 | 8·49 | 7·46 | 6·81 | 6·35 | 5·76 | 5·13 | 4·45 | 3·67 |
| 19 | 15·08 | 10·16 | 8·28 | 7·26 | 6·62 | 6·18 | 5·59 | 4·97 | 4·29 | 3·52 |
| 20 | 14·82 | 9·95 | 8·10 | 7·10 | 6·46 | 6·02 | 5·44 | 4·82 | 4·15 | 3·38 |
| 21 | 14·59 | 9·77 | 7·94 | 6·95 | 6·32 | 5·88 | 5·31 | 4·70 | 4·03 | 3·26 |
| 22 | 14·38 | 9·61 | 7·80 | 6·81 | 6·19 | 5·76 | 5·19 | 4·58 | 3·92 | 3·15 |
| 23 | 14·19 | 9·47 | 7·67 | 6·69 | 6·08 | 5·65 | 5·09 | 4·48 | 3·82 | 3·05 |
| 24 | 14·03 | 9·34 | 7·55 | 6·59 | 5·98 | 5·55 | 4·99 | 4·39 | 3·74 | 2·97 |
| 25 | 13·88 | 9·22 | 7·45 | 6·49 | 5·88 | 5·46 | 4·91 | 4·31 | 3·66 | 2·89 |
| 26 | 13·74 | 9·12 | 7·36 | 6·41 | 5·80 | 5·38 | 4·83 | 4·24 | 3·59 | 2·82 |
| 27 | 13·61 | 9·02 | 7·27 | 6·33 | 5·73 | 5·31 | 4·76 | 4·17 | 3·52 | 2·75 |
| 28 | 13·50 | 8·93 | 7·19 | 6·25 | 5·66 | 5·24 | 4·69 | 4·11 | 3·46 | 2·70 |
| 29 | 13·39 | 8·85 | 7·12 | 6·19 | 5·59 | 5·18 | 4·64 | 4·05 | 3·41 | 2·64 |
| 30 | 13·29 | 8·77 | 7·05 | 6·12 | 5·53 | 5·12 | 4·58 | 4·00 | 3·36 | 2·59 |
| 40 | 12·61 | 8·25 | 6·60 | 5·70 | 5·13 | 4·73 | 4·21 | 3·64 | 3·01 | 2·23 |
| 60 | 11·97 | 7·76 | 6·17 | 5·31 | 4·76 | 4·37 | 3·87 | 3·31 | 2·69 | 1·90 |
| 120 | 11·38 | 7·32 | 5·79 | 4·95 | 4·42 | 4·04 | 3·55 | 3·02 | 2·40 | 1·54 |
| ∞ | 10·83 | 6·91 | 5·42 | 4·62 | 4·10 | 3·74 | 3·27 | 2·74 | 2·13 | 1·00 |

Lower 0·1 per cent. points are found by interchange of $n_1$ and $n_2$, *i.e.* $n_1$ must always correspond with the greater mean square.

# TABLE VI. SIGNIFICANCE OF DIFFERENCE BETWEEN TWO MEANS

(P. V. Sukhatme)

| | $n_1$ | 0° | 15° | 30° | 45° | 60° | 75° | 90° |
|---|---|---|---|---|---|---|---|---|
| 5 per cent. points<br>$n_2 = 6$ | 6 | 2·447 | 2·440 | 2·435 | 2·435 | 2·435 | 2·440 | 2·447 |
| | 8 | 2·447 | 2·430 | 2·398 | 2·364 | 2·331 | 2·310 | 2·306 |
| | 12 | 2·447 | 2·423 | 2·367 | 2·301 | 2·239 | 2·193 | 2·179 |
| | 24 | 2·447 | 2·418 | 2·342 | 2·247 | 2·156 | 2·088 | 2·064 |
| | ∞ | 2·447 | 2·413 | 2·322 | 2·201 | 2·082 | 1·993 | 1·960 |
| $n_2 = 8$ | 6 | 2·306 | 2·310 | 2·331 | 2·364 | 2·398 | 2·430 | 2·447 |
| | 8 | 2·306 | 2·300 | 2·294 | 2·292 | 2·294 | 2·300 | 2·306 |
| | 12 | 2·306 | 2·292 | 2·262 | 2·229 | 2·201 | 2·183 | 2·179 |
| | 24 | 2·306 | 2·286 | 2·236 | 2·175 | 2·118 | 2·077 | 2·064 |
| | ∞ | 2·306 | 2·281 | 2·215 | 2·128 | 2·044 | 1·982 | 1·960 |
| $n_2 = 12$ | 6 | 2·179 | 2 193 | 2·239 | 2·301 | 2·367 | 2·423 | 2·447 |
| | 8 | 2·179 | 2 183 | 2·201 | 2·229 | 2·262 | 2·292 | 2·306 |
| | 12 | 2·179 | 2·175 | 2·169 | 2·167 | 2·169 | 2·175 | 2·179 |
| | 24 | 2·179 | 2·168 | 2·142 | 2·112 | 2·085 | 2·069 | 2·064 |
| | ∞ | 2·179 | 2·163 | 2·120 | 2·064 | 2·011 | 1·973 | 1·960 |
| $n_2 = 24$ | 6 | 2·064 | 2·088 | 2·156 | 2·247 | 2·342 | 2·418 | 2·447 |
| | 8 | 2·064 | 2·077 | 2·118 | 2·175 | 2·236 | 2·286 | 2·306 |
| | 12 | 2·064 | 2·069 | 2·085 | 2·112 | 2·142 | 2·168 | 2·179 |
| | 24 | 2·064 | 2·062 | 2·058 | 2·056 | 2·058 | 2·062 | 2·064 |
| | ∞ | 2·064 | 2·056 | 2·035 | 2·009 | 1·983 | 1·966 | 1·960 |
| $n_2 = \infty$ | 6 | 1·960 | 1·993 | 2·082 | 2·201 | 2·322 | 2·413 | 2·447 |
| | 8 | 1·960 | 1·982 | 2·044 | 2·128 | 2·215 | 2·281 | 2·306 |
| | 12 | 1·960 | 1·973 | 2·011 | 2·064 | 2·120 | 2·163 | 2·179 |
| | 24 | 1·960 | 1·966 | 1·983 | 2·009 | 2·035 | 2·056 | 2·064 |
| | ∞ | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 |
| 1 per cent. points<br>$n_2 = 6$ | 6 | 3·707 | 3·654 | 3·557 | 3·514 | 3·557 | 3·654 | 3·707 |
| | 8 | 3·707 | 3·643 | 3·495 | 3·363 | 3·307 | 3·328 | 3·355 |
| | 12 | 3·707 | 3·636 | 3·453 | 3·246 | 3·104 | 3·053 | 3·055 |
| | 24 | 3·707 | 3·631 | 3·424 | 3·158 | 2·938 | 2·822 | 2·797 |
| | ∞ | 3·707 | 3·626 | 3·402 | 3·093 | 2·804 | 2·627 | 2·576 |
| $n_2 = 8$ | 6 | 3·355 | 3·328 | 3·307 | 3·363 | 3·495 | 3·643 | 3·707 |
| | 8 | 3·355 | 3·316 | 3·239 | 3·206 | 3·239 | 3·316 | 3·355 |
| | 12 | 3·355 | 3·307 | 3·192 | 3·083 | 3·032 | 3·039 | 3·055 |
| | 24 | 3·355 | 3·301 | 3·158 | 2·988 | 2·862 | 2·805 | 2·797 |
| | ∞ | 3·355 | 3·295 | 3·132 | 2·916 | 2·723 | 2·608 | 2·576 |
| $n_2 = 12$ | 6 | 3·055 | 3·053 | 3·104 | 3·246 | 3·453 | 3·636 | 3·707 |
| | 8 | 3·055 | 3·039 | 3·032 | 3·083 | 3·192 | 3·307 | 3·355 |
| | 12 | 3·055 | 3·029 | 2·978 | 2·954 | 2·978 | 3·029 | 3·055 |
| | 24 | 3·055 | 3·020 | 2·938 | 2·853 | 2·803 | 2·793 | 2·797 |
| | ∞ | 3·055 | 3·014 | 2·909 | 2·775 | 2·661 | 2·595 | 2·576 |
| $n_2 = 24$ | 6 | 2·797 | 2·822 | 2·938 | 3·158 | 3·424 | 3·631 | 3·707 |
| | 8 | 2·797 | 2·805 | 2·862 | 2·988 | 3·158 | 3·301 | 3·355 |
| | 12 | 2·797 | 2·793 | 2·803 | 2·853 | 2·938 | 3·020 | 3·055 |
| | 24 | 2·797 | 2·785 | 2·759 | 2·747 | 2·759 | 2·785 | 2·797 |
| | ∞ | 2·797 | 2·777 | 2·726 | 2·664 | 2·613 | 2·585 | 2·576 |
| $n_2 = \infty$ | 6 | 2·576 | 2·627 | 2·804 | 3·093 | 3·402 | 3·626 | 3·707 |
| | 8 | 2·576 | 2·608 | 2·723 | 2·916 | 3·132 | 3·295 | 3·355 |
| | 12 | 2·576 | 2·595 | 2·661 | 2·775 | 2·909 | 3·014 | 3·055 |
| | 24 | 2·576 | 2·585 | 2·613 | 2·664 | 2·726 | 2·777 | 2·797 |
| | ∞ | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 |

## TABLE V2. SIGNIFICANCE OF DIFFERENCE BETWEEN TWO MEANS

One component of error distributed normally, the other in Student's distribution

| $P$ | $n_2$ | 0°($t$) | 10° | 20° | 30° | 40° | 50° | 60° | 70° | 80° | 90°($x$) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 per cent. | 10 | 1·812 | 1·808 | 1·794 | 1·774 | 1·749 | 1·721 | 1·693 | 1·668 | 1·651 | 1·645 |
| | 12 | 1·782 | 1·778 | 1·767 | 1·751 | 1·730 | 1·707 | 1·684 | 1·664 | 1·650 | 1·645 |
| | 15 | 1·753 | 1·750 | 1·741 | 1·728 | 1·711 | 1·693 | 1·675 | 1·659 | 1·649 | 1·645 |
| | 20 | 1·725 | 1·722 | 1·716 | 1·706 | 1·694 | 1·680 | 1·667 | 1·656 | 1·648 | 1·645 |
| | 30 | 1·697 | 1·696 | 1·692 | 1·685 | 1·677 | 1·668 | 1·659 | 1·652 | 1·647 | 1·645 |
| | 60 | 1·671 | 1·670 | 1·668 | 1·665 | 1·661 | 1·656 | 1·652 | 1·648 | 1·646 | 1·645 |
| | $\infty$ | 1·645 | 1·645 | 1·645 | 1·645 | 1·645 | 1·645 | 1·645 | 1·645 | 1·645 | 1·645 |
| 5 per cent. | 10 | 2·228 | 2·219 | 2·194 | 2·157 | 2·112 | 2·066 | 2·024 | 1·989 | 1·967 | 1·960 |
| | 12 | 2·179 | 2·171 | 2·151 | 2·120 | 2·083 | 2·046 | 2·011 | 1·984 | 1·966 | 1·960 |
| | 15 | 2·131 | 2·126 | 2·109 | 2·085 | 2·056 | 2·026 | 1·999 | 1·978 | 1·965 | 1·960 |
| | 20 | 2·086 | 2·082 | 2·069 | 2·051 | 2·030 | 2·008 | 1·989 | 1·973 | 1·963 | 1·960 |
| | 30 | 2·042 | 2·039 | 2·031 | 2·019 | 2·005 | 1·991 | 1·978 | 1·968 | 1·962 | 1·960 |
| | 60 | 2·000 | 1·999 | 1·995 | 1·989 | 1·982 | 1·975 | 1·969 | 1·964 | 1·961 | 1·960 |
| | $\infty$ | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 | 1·960 |
| 2 per cent. | 10 | 2·764 | 2·748 | 2·704 | 2·637 | 2·559 | 2·481 | 2·414 | 2·364 | 2·335 | 2·326 |
| | 12 | 2·681 | 2·668 | 2·631 | 2·576 | 2·513 | 2·450 | 2·396 | 2·356 | 2·334 | 2·326 |
| | 15 | 2·602 | 2·592 | 2·563 | 2·520 | 2·470 | 2·421 | 2·379 | 2·349 | 2·332 | 2·326 |
| | 20 | 2·528 | 2·520 | 2·498 | 2·466 | 2·430 | 2·394 | 2·364 | 2·343 | 2·330 | 2·326 |
| | 30 | 2·457 | 2·452 | 2·438 | 2·417 | 2·393 | 2·370 | 2·351 | 2·337 | 2·329 | 2·326 |
| | 60 | 2·390 | 2·388 | 2·380 | 2·370 | 2·358 | 2·347 | 2·338 | 2·331 | 2·328 | 2·326 |
| | $\infty$ | 2·326 | 2·326 | 2·326 | 2·326 | 2·326 | 2·326 | 2·326 | 2·326 | 2·326 | 2·326 |
| 1 per cent. | 10 | 3·169 | 3·148 | 3·086 | 2·993 | 2·883 | 2·775 | 2·684 | 2·620 | 2·586 | 2·576 |
| | 12 | 3·055 | 3·037 | 2·985 | 2·909 | 2·820 | 2·733 | 2·661 | 2·611 | 2·584 | 2·576 |
| | 15 | 2·947 | 2·932 | 2·892 | 2·831 | 2·762 | 2·695 | 2·640 | 2·603 | 2·582 | 2·576 |
| | 20 | 2·845 | 2·835 | 2·804 | 2·760 | 2·709 | 2·661 | 2·622 | 2·595 | 2·580 | 2·576 |
| | 30 | 2·750 | 2·743 | 2·723 | 2·693 | 2·661 | 2·630 | 2·605 | 2·588 | 2·579 | 2·576 |
| | 60 | 2·660 | 2·657 | 2·647 | 2·632 | 2·616 | 2·601 | 2·590 | 2·582 | 2·577 | 2·576 |
| | $\infty$ | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 | 2·576 |
| 0·5 per cent. | 10 | 3·581 | 3·553 | 3·473 | 3·350 | 3·203 | 3·058 | 2·939 | 2·859 | 2·818 | 2·807 |
| | 12 | 3·429 | 3·405 | 3·338 | 3·237 | 3·119 | 3·003 | 2·910 | 2·848 | 2·816 | 2·807 |
| | 15 | 3·286 | 3·267 | 3·214 | 3·134 | 3·042 | 2·954 | 2·884 | 2·838 | 2·814 | 2·807 |
| | 20 | 3·153 | 3·139 | 3·099 | 3·040 | 2·974 | 2·911 | 2·861 | 2·829 | 2·812 | 2·807 |
| | 30 | 3·030 | 3·020 | 2·994 | 2·955 | 2·912 | 2·872 | 2·841 | 2·821 | 2·810 | 2·807 |
| | 60 | 2·915 | 2·910 | 2·897 | 2·878 | 2·857 | 2·838 | 2·823 | 2·814 | 2·809 | 2·807 |
| | $\infty$ | 2·807 | 2·807 | 2·807 | 2·807 | 2·807 | 2·807 | 2·807 | 2·807 | 2·807 | 2·807 |
| 0·2 per cent. | 10 | 4·144 | 4·106 | 3·999 | 3·832 | 3·630 | 3·425 | 3·259 | 3·152 | 3·103 | 3·090 |
| | 12 | 3·930 | 3·898 | 3·809 | 3·671 | 3·508 | 3·347 | 3·219 | 3·138 | 3·100 | 3·090 |
| | 15 | 3·733 | 3·708 | 3·636 | 3·528 | 3·401 | 3·280 | 3·185 | 3·126 | 3·098 | 3·090 |
| | 20 | 3·552 | 3·533 | 3·479 | 3·399 | 3·308 | 3·222 | 3·156 | 3·116 | 3·096 | 3·090 |
| | 30 | 3·386 | 3·372 | 3·336 | 3·284 | 3·226 | 3·172 | 3·131 | 3·106 | 3·094 | 3·090 |
| | 60 | 3·232 | 3·225 | 3·207 | 3·181 | 3·153 | 3·128 | 3·110 | 3·098 | 3·092 | 3·090 |
| | $\infty$ | 3·090 | 3·090 | 3·090 | 3·090 | 3·090 | 3·090 | 3·090 | 3·090 | 3·090 | 3·090 |

TABLE VI. Values of the Correlation Coefficient for Different Levels of Significance

| *n* | ·1 | ·05 | ·02 | ·01 | ·001 | *n* | ·1 | ·05 | ·02 | ·01 | ·001 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | ·98769 | ·99692 | ·999507 | ·999877 | ·9999988 | 16 | ·4000 | ·4683 | ·5425 | ·5897 | ·7084 |
| 2 | ·90000 | ·95000 | ·98000 | ·990000 | ·99900 | 17 | ·3887 | ·4555 | ·5285 | ·5751 | ·6932 |
| 3 | ·8054 | ·8783 | ·93433 | ·95873 | ·99116 | 18 | ·3783 | ·4438 | ·5155 | ·5614 | ·6787 |
| 4 | ·7293 | ·8114 | ·8822 | ·91720 | ·97406 | 19 | ·3687 | ·4329 | ·5034 | ·5487 | ·6652 |
| 5 | ·6694 | ·7545 | ·8329 | ·8745 | ·95074 | 20 | ·3598 | ·4227 | ·4921 | ·5368 | ·6524 |
| 6 | ·6215 | ·7067 | ·7887 | ·8343 | ·92493 | 25 | ·3233 | ·3809 | ·4451 | ·4869 | ·5974 |
| 7 | ·5822 | ·6664 | ·7498 | ·7977 | ·8982 | 30 | ·2960 | ·3494 | ·4093 | ·4487 | ·5541 |
| 8 | ·5494 | ·6319 | ·7155 | ·7646 | ·8721 | 35 | ·2746 | ·3246 | ·3810 | ·4182 | ·5189 |
| 9 | ·5214 | ·6021 | ·6851 | ·7348 | ·8471 | 40 | ·2573 | ·3044 | ·3578 | ·3932 | ·4896 |
| 10 | ·4973 | ·5760 | ·6581 | ·7079 | ·8233 | 45 | ·2428 | ·2875 | ·3384 | ·3721 | ·4648 |
| 11 | ·4762 | ·5529 | ·6339 | ·6835 | ·8010 | 50 | ·2306 | ·2732 | ·3218 | ·3541 | ·4433 |
| 12 | ·4575 | ·5324 | ·6120 | ·6614 | ·7800 | 60 | ·2108 | ·2500 | ·2948 | ·3248 | ·4078 |
| 13 | ·4409 | ·5139 | ·5923 | ·6411 | ·7603 | 70 | ·1954 | ·2319 | ·2737 | ·3017 | ·3799 |
| 14 | ·4259 | ·4973 | ·5742 | ·6226 | ·7420 | 80 | ·1829 | ·2172 | ·2565 | ·2830 | ·3568 |
| 15 | ·4124 | ·4821 | ·5577 | ·6055 | ·7246 | 90 | ·1726 | ·2050 | ·2422 | ·2673 | ·3375 |
|  |  |  |  |  |  | 100 | ·1638 | ·1946 | ·2301 | ·2540 | ·3211 |

TABLE VII. Transformation of *r* to *z* (Interclass)

| *z* | ·00 | ·01 | ·02 | ·03 | ·04 | ·05 | ·06 | ·07 | ·08 | ·09 | Mean Diff. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ·0 | ·0000 | ·0100 | ·0200 | ·0300 | ·0400 | ·0500 | ·0599 | ·0699 | ·0798 | ·0898 | 100 |
| ·1 | ·0997 | ·1096 | ·1194 | ·1293 | ·1391 | ·1489 | ·1586 | ·1684 | ·1781 | ·1877 | 98 |
| ·2 | ·1974 | ·2070 | ·2165 | ·2260 | ·2355 | ·2449 | ·2543 | ·2636 | ·2729 | ·2821 | 94 |
| ·3 | ·2913 | ·3004 | ·3095 | ·3185 | ·3275 | ·3364 | ·3452 | ·3540 | ·3627 | ·3714 | 89 |
| ·4 | ·3800 | ·3885 | ·3969 | ·4053 | ·4136 | ·4219 | ·4301 | ·4382 | ·4462 | ·4542 | 82 |
| ·5 | ·4621 | ·4699 | ·4777 | ·4854 | ·4930 | ·5005 | ·5080 | ·5154 | ·5227 | ·5299 | 75 |
| ·6 | ·5370 | ·5441 | ·5511 | ·5580 | ·5649 | ·5717 | ·5784 | ·5850 | ·5915 | ·5980 | 68 |
| ·7 | ·6044 | ·6107 | ·6169 | ·6231 | ·6291 | ·6351 | ·6411 | ·6469 | ·6527 | ·6584 | 60 |
| ·8 | ·6640 | ·6696 | ·6751 | ·6805 | ·6858 | ·6911 | ·6963 | ·7014 | ·7064 | ·7114 | 53 |
| ·9 | ·7163 | ·7211 | ·7259 | ·7306 | ·7352 | ·7398 | ·7443 | ·7487 | ·7531 | ·7574 | 46 |
| 1·0 | ·7616 | ·7658 | ·7699 | ·7739 | ·7779 | ·7818 | ·7857 | ·7895 | ·7932 | ·7969 | 39 |
| 1·1 | ·8005 | ·8041 | ·8076 | ·8110 | ·8144 | ·8178 | ·8210 | ·8243 | ·8275 | ·8306 | 33 |
| 1·2 | ·8337 | ·8367 | ·8397 | ·8426 | ·8455 | ·8483 | ·8511 | ·8538 | ·8565 | ·8591 | 28 |
| 1·3 | ·8617 | ·8643 | ·8668 | ·8692 | ·8717 | ·8741 | ·8764 | ·8787 | ·8810 | ·8832 | 24 |
| 1·4 | ·8854 | ·8875 | ·8896 | ·8917 | ·8937 | ·8957 | ·8977 | ·8996 | ·9015 | ·9033 | 20 |
| 1·5 | ·9051 | ·9069 | ·9087 | ·9104 | ·9121 | ·9138 | ·9154 | ·9170 | ·9186 | ·9201 | 17 |
| 1·6 | ·9217 | ·9232 | ·9246 | ·9261 | ·9275 | ·9289 | ·9302 | ·9316 | ·9329 | ·9341 | 14 |
| 1·7 | ·9354 | ·9366 | ·9379 | ·9391 | ·9402 | ·9414 | ·9425 | ·9436 | ·9447 | ·9458 | 12 |
| 1·8 | ·94681 | ·94783 | ·94884 | ·94983 | ·95080 | ·95175 | ·95268 | ·95359 | ·95449 | ·95537 | 95 |
| 1·9 | ·95624 | ·95709 | ·95792 | ·95873 | ·95953 | ·96032 | ·96109 | ·96185 | ·96259 | ·96331 | 79 |
| 2·0 | ·96403 | ·96473 | ·96541 | ·96609 | ·96675 | ·96739 | ·96803 | ·96865 | ·96926 | ·96986 | 65 |
| 2·1 | ·97045 | ·97103 | ·97159 | ·97215 | ·97269 | ·97323 | ·97375 | ·97426 | ·97477 | ·97526 | 53 |
| 2·2 | ·97574 | ·97622 | ·97668 | ·97714 | ·97759 | ·97803 | ·97846 | ·97888 | ·97929 | ·97970 | 44 |
| 2·3 | ·98010 | ·98049 | ·98087 | ·98124 | ·98161 | ·98197 | ·98233 | ·98267 | ·98301 | ·98335 | 36 |
| 2·4 | ·98367 | ·98399 | ·98431 | ·98462 | ·98492 | ·98522 | ·98551 | ·98579 | ·98607 | ·98635 | 30 |
| 2·5 | ·98661 | ·98688 | ·98714 | ·98739 | ·98764 | ·98788 | ·98812 | ·98835 | ·98858 | ·98881 | 24 |
| 2·6 | ·98903 | ·98924 | ·98945 | ·98966 | ·98987 | ·99007 | ·99026 | ·99045 | ·99064 | ·99083 | 20 |
| 2·7 | ·99101 | ·99118 | ·99136 | ·99153 | ·99170 | ·99186 | ·99202 | ·99218 | ·99233 | ·99248 | 16 |
| 2·8 | ·99263 | ·99278 | ·99292 | ·99306 | ·99320 | ·99333 | ·99346 | ·99359 | ·99372 | ·99384 | 13 |
| 2·9 | ·99396 | ·99408 | ·99420 | ·99431 | ·99443 | ·99454 | ·99464 | ·99475 | ·99485 | ·99495 | 11 |
|  | ·0 | ·1 | ·2 | ·3 | ·4 | ·5 | ·6 | ·7 | ·8 | ·9 |  |
| 3 | ·99505 | ·99595 | ·99668 | ·99728 | ·99777 | ·99818 | ·99851 | ·99878 | ·99900 | ·99918 | — |
| 4 | ·99933 | ·99945 | ·99955 | ·99963 | ·99970 | ·99975 | ·99980 | ·99983 | ·99986 | ·99989 | — |

For notes see foot of opposite page.

# TABLE VIII. TESTS OF SIGNIFICANCE FOR 2×2 CONTINGENCY TABLES

Calculate $\chi_c$, the square root of $\chi^2$ corrected for continuity (Fisher, *Statistical Methods*, 21.01). Determine $m$ = the smallest expectation of any class, and $p = \dfrac{\text{the smallest expectation}}{\text{the smallest marginal total}}$.

The table gives the 2·5 and 0·5 per cent. points of $\chi_c$ for each tail separately, these being different for the two tails except when $p = 0{\cdot}5$. The observed set of values will lie on the longer tail if the observed number in the cell with smallest expectation is greater than expectation.

Since three of the four marginal totals are independent their values are not completely determined by $m$ and $p$. For a given $m$ and $p$ there is a whole set of contingency distributions, and a corresponding set of values of $\chi_c$ for each level of significance. The variation of these $\chi_c$ is, however, small. The table gives the greatest and least values, *i.e.* those for the binomial distribution (Roman type), and the limiting contingency distribution (black type). When $p = 0$ there is only one value, this being derived from the Poisson distribution.

The table should always be used in place of the ordinary $\chi^2$ table when $m$ is less than 10. Even with values of $m$ as great as 100 the gain in precision is appreciable when $p$ is small and the interest centres in the probability of a single tail (*i.e.* deviations in one direction only). The correction for continuity should be used when $m$ is less than 500.

In regions of the table where no values are given, and to determine the exact probability in doubtful cases, the exact solution (on which this table is based) must be used (Fisher, *Statistical Methods*, 21.02).

The table can be used for testing the deviation of a sample of a binomial distribution with known $p$. Remember to include the contributions to $\chi^2$ from both classes when calculating $\chi_c$.

| | $p$ / $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | 48 | 96 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **P = ·025** . . | 0 | ... | ... | ... | 1·68 | 1·71 | 1·74 | 1·77 | 1·81 | 1·86 | 1·89 | 1·91 |
| Shorter tail . . | 0·25 | ... | ... | 1·73 | 1·77 | 1·80 | 1·82 | 1·84 | 1·87 | 1·90 | 1·91 | 1·93 |
| | | ... | ... | **1·83** | **1·85** | **1·86** | **1·87** | **1·89** | **1·90** | **1·92** | **1·94** | **1·94** |
| | 0·5 | ... | ... | 1·88 | 1·90 | 1·91 | 1·92 | 1·93 | 1·94 | 1·95 | 1·96 | 1·96 |
| | | ... | **1·91** | **1·93** | **1·94** | **1·94** | **1·94** | **1·95** | **1·95** | **1·95** | **1·96** | **1·96** |
| Longer tail . . | 0·25 | 2·10 | 2·08 | 2·07 | 2·06 | 2·06 | 2·05 | 2·04 | 2·02 | 2·01 | 2·00 | 1·99 |
| | | **2·04** | **2·03** | **2·02** | **2·02** | **2·01** | **2·01** | **2·00** | **1·99** | **1·99** | **1·98** | **1·97** |
| | 0 | 2·32 | 2·24 | 2·19 | 2·16 | 2·14 | 2·13 | 2·11 | 2·08 | 2·05 | 2·03 | 2·01 |
| **P = ·005** . . | 0 | ... | ... | ... | ... | 2·06 | 2·13 | 2·19 | 2·27 | 2·37 | 2·43 | 2·48 |
| Shorter tail . . | 0·25 | ... | ... | ... | 2·18 | 2·23 | 2·27 | 2·32 | 2·38 | 2·45 | 2·49 | 2·52 |
| | | ... | ... | ... | **2·32** | **2·36** | **2·38** | **2·42** | **2·46** | **2·50** | **2·52** | **2·54** |
| | 0·5 | ... | ... | ... | 2·41 | 2·44 | 2·47 | 2·50 | 2·52 | 2·55 | 2·56 | 2·57 |
| | | ... | ... | **2·48** | **2·50** | **2·52** | **2·53** | **2·54** | **2·55** | **2·56** | **2·57** | **2·57** |
| Longer tail . . | 0·25 | 2·79 | 2·79 | 2·78 | 2·76 | 2·75 | 2·73 | 2·72 | 2·70 | 2·67 | 2·64 | 2·63 |
| | | **2·67** | **2·67** | **2·67** | **2·67** | **2·66** | **2·66** | **2·65** | **2·64** | **2·63** | **2·62** | **2·60** |
| | 0 | 3·33 | 3·13 | 3·05 | 2·97 | 2·95 | 2·92 | 2·88 | 2·83 | 2·76 | 2·70 | 2·67 |

Notes on Tables VI and VII.

For a total correlation, $n$ is 2 less than the number of pairs in the sample; for a partial correlation, the number of eliminated variates also should be subtracted. The probability given at the head of each column of Table VI represents the chance that $r$ will be greater than the values given or less than *minus* these values.

Table VII gives the transformation $r = (e^{2z}-1)/(e^{2z}+1)$ or $z = \frac{1}{2}\{\log_e(1+r)-\log_e(1-r)\}$. With $n$ defined as above $z$ is distributed approximately normally with variance $1/(n-1)$. For exact work correct for bias in $z$ by subtracting $r/2(n+1)$ from $z$.

The table is also sometimes useful for transforming the probability scale, being equivalent to $z = \frac{1}{2}\{\log p - \log(1-p)\}$ where $r = 2p-1$. (See Introduction).

# Table VIII$_1$. Binomial and Poisson Distributions: Limits of the Expectation

(Based on W. L. Stevens)

| $a$ | $N$ | $p = a/N$ | Probability $P$ of $a$ or more | | | Probability $P$ of $a$ or fewer | | |
|---|---|---|---|---|---|---|---|---|
| | | | ·005 | ·025 | ·1 | ·1 | ·025 | ·005 |
| 0 | 5 | (·2) | For $a = 0$, $1-P^{1/N}$ is the greatest probability allowable. | | | 1·84 | 2·61 | 3·27 |
| | | (·15) | | | | 1·95 | 2·83 | 3·66 |
| | 10 | (·1) | | | | 2·06 | 3·09 | 4·11 |
| | 20 | (·05) | | | | 2·17 | 3·37 | 4·65 |
| | ∞ | 0 | | | | 2·30 | 3·69 | 5·30 |
| 1 | 2 | | ·0050 | ·0252 | ·103 | 1·897 | 1·9748 | 1·9950 |
| | 3 | | ·0050 | ·0252 | ·104 | 2·413 | 2·7171 | 2·8758 |
| | 4 | | ·0050 | ·0252 | ·104 | 2·718 | 3·224 | 3·556 |
| | 5 | ·2 | ·0050 | ·0253 | ·104 | 2·92 | 3·58 | 4·07 |
| | | ·15 | ·0050 | ·0253 | ·105 | 3·14 | 3·99 | 4·69 |
| | 10 | ·1 | ·0050 | ·0253 | ·105 | 3·37 | 4·45 | 5·44 |
| | 20 | ·05 | ·0050 | ·0253 | ·105 | 3·62 | 4·97 | 6·34 |
| | ∞ | 0 | ·0050 | ·0253 | ·105 | 3·89 | 5·57 | 7·43 |
| 2 | 4 | | ·118 | ·270 | ·570 | 3·430 | 3·730 | 3·882 |
| | 5 | | ·114 | ·264 | ·561 | 3·77 | 4·267 | 4·586 |
| | 6 | | ·112 | ·260 | ·556 | 4·00 | 4·67 | 5·138 |
| | 7 | | ·111 | ·257 | ·552 | 4·17 | 4·97 | 5·54 |
| | 8 | ·25 | ·110 | ·255 | ·549 | 4·31 | 5·21 | 5·94 |
| | 10 | ·2 | ·108 | ·252 | ·545 | 4·50 | 5·56 | 6·48 |
| | | ·15 | ·107 | ·249 | ·542 | 4·69 | 5·94 | 7·09 |
| | 20 | ·1 | ·106 | ·247 | ·538 | 4·90 | 6·34 | 7·74 |
| | 40 | ·05 | ·105 | ·245 | ·535 | 5·11 | 6·77 | 8·44 |
| | ∞ | 0 | ·103 | ·242 | ·532 | 5·32 | 7·22 | 9·27 |
| 3 | 6 | | ·398 | ·709 | 1·21 | 4·79 | 5·291 | 5·602 |
| | 7 | | ·387 | ·693 | 1·19 | 5·05 | 5·71 | 6·181 |
| | 8 | | ·380 | ·682 | 1·17 | 5·24 | 6·04 | 6·642 |
| | 9 | | ·374 | ·674 | 1·17 | 5·39 | 6·31 | 7·03 |
| | 10 | ·3 | ·370 | ·667 | 1·16 | 5·52 | 6·52 | 7·35 |
| | 15 | ·2 | ·358 | ·650 | 1·14 | 5·89 | 7·21 | 8·41 |
| | 30 | ·1 | ·348 | ·634 | 1·12 | 6·28 | 7·96 | 9·61 |
| | ∞ | 0 | ·338 | ·619 | 1·10 | 6·68 | 8·77 | 10·98 |
| 4 | 8 | ·5 | ·799 | 1·26 | 1·92 | 6·08 | 6·74 | 7·201 |
| | 10 | ·4 | ·768 | 1·22 | 1·88 | 6·46 | 7·38 | 8·091 |
| | | ·3 | ·741 | 1·18 | 1·84 | 6·83 | 8·04 | 9·06 |
| | 20 | ·2 | ·716 | 1·15 | 1·80 | 7·21 | 8·73 | 10·13 |
| | 40 | ·1 | ·694 | 1·12 | 1·77 | 7·60 | 9·47 | 11·31 |
| | ∞ | 0 | ·672 | 1·09 | 1·74 | 7·99 | 10·24 | 12·59 |
| 5 | 10 | ·5 | 1·28 | 1·87 | 2·67 | 7·33 | 8·13 | 8·72 |
| | | ·4 | 1·23 | 1·81 | 2·61 | 7·72 | 8·79 | 9·66 |
| | | ·3 | 1·19 | 1·76 | 2·56 | 8·10 | 9·47 | 10·67 |
| | 25 | ·2 | 1·15 | 1·71 | 2·52 | 8·49 | 10·18 | 11·75 |
| | 50 | ·1 | 1·11 | 1·66 | 2·47 | 8·88 | 10·91 | 12·90 |
| | ∞ | 0 | 1·08 | 1·62 | 2·43 | 9·27 | 11·67 | 14·15 |
| 6 | 12 | ·5 | 1·83 | 2·53 | 3·46 | 8·54 | 9·47 | 10·17 |
| | 15 | ·4 | 1·75 | 2·45 | 3·38 | 8·95 | 10·16 | 11·16 |
| | 20 | ·3 | 1·69 | 2·38 | 3·32 | 9·35 | 10·86 | 12·19 |
| | 30 | ·2 | 1·63 | 2·31 | 3·26 | 9·74 | 11·57 | 13·28 |
| | 60 | ·1 | 1·58 | 2·26 | 3·20 | 10·14 | 12·30 | 14·44 |
| | ∞ | 0 | 1·54 | 2·20 | 3·15 | 10·53 | 13·06 | 15·66 |
| 7 | 14 | ·5 | 2·41 | 3·23 | 4·26 | 9·74 | 10·77 | 11·59 |
| | | ·4 | 2·32 | 3·12 | 4·17 | 10·15 | 11·48 | 12·61 |
| | | ·3 | 2·24 | 3·03 | 4·09 | 10·56 | 12·20 | 13·67 |
| | 35 | ·2 | 2·16 | 2·95 | 4·02 | 10·97 | 12·93 | 14·77 |
| | 70 | ·1 | 2·10 | 2·88 | 3·96 | 11·37 | 13·67 | 15·92 |
| | ∞ | 0 | 2·04 | 2·81 | 3·89 | 11·77 | 14·42 | 17·13 |

| $a$ | $p = a/N$ | Probability $P$ of $a$ or more | | | Probability $P$ of $a$ or fewer | | |
|---|---|---|---|---|---|---|---|
| | | ·005 | ·025 | ·1 | ·1 | ·025 | ·005 |
| 8 | ·5 | 3·04 | 3·94 | 5·09 | 10·91 | 12·06 | 12·96 |
| | ·4 | 2·92 | 3·82 | 4·98 | 11·35 | 12·79 | 14·02 |
| | ·3 | 2·82 | 3·72 | 4·89 | 11·77 | 13·52 | 15·10 |
| | ·2 | 2·73 | 3·62 | 4·80 | 12·18 | 14·26 | 16·22 |
| | ·1 | 2·65 | 3·53 | 4·73 | 12·59 | 15·01 | 17·37 |
| | 0 | 2·57 | 3·45 | 4·66 | 12·99 | 15·76 | 18·58 |
| 9 | ·5 | 3·68 | 4·68 | 5·92 | 12·08 | 13·32 | 14·32 |
| | ·4 | 3·55 | 4·54 | 5·80 | 12·52 | 14·07 | 15·40 |
| | ·3 | 3·43 | 4·42 | 5·70 | 12·96 | 14·82 | 16·50 |
| | ·2 | 3·32 | 4·31 | 5·60 | 13·38 | 15·57 | 17·63 |
| | ·1 | 3·22 | 4·21 | 5·51 | 13·79 | 16·32 | 18·80 |
| | 0 | 3·13 | 4·12 | 5·43 | 14·21 | 17·08 | 20·00 |
| 10 | ·5 | 4·35 | 5·44 | 6·76 | 13·24 | 14·56 | 15·65 |
| | ·4 | 4·20 | 5·28 | 6·63 | 13·69 | 15·33 | 16·76 |
| | ·3 | 4·06 | 5·14 | 6·51 | 14·13 | 16·10 | 17·88 |
| | ·2 | 3·93 | 5·02 | 6·41 | 14·56 | 16·86 | 19·02 |
| | ·1 | 3·82 | 4·90 | 6·31 | 14·99 | 17·62 | 20·20 |
| | 0 | 3·72 | 4·80 | 6·22 | 15·41 | 18·39 | 21·40 |
| 9 | ·5 | +0·14 | −0·16 | −0·36 | +0·36 | +0·16 | −0·14 |
| | ·4 | +0·53 | +0·10 | −0·22 | +0·55 | +0·52 | +0·41 |
| | ·3 | +0·89 | +0·34 | −0·09 | +0·74 | +0·90 | +1·04 |
| | ·2 | +1·23 | +0·57 | +0·04 | +0·94 | +1·31 | +1·72 |
| | ·1 | +1·55 | +0·79 | +0·16 | +1·15 | +1·74 | +2·47 |
| | 0 | +1·86 | +1·00 | +0·28 | +1·36 | +2·20 | +3·27 |
| 16 | ·5 | −0·02 | −0·25 | −0·40 | +0·40 | +0·25 | +0·02 |
| | ·4 | +0·40 | +0·02 | −0·26 | +0·58 | +0·60 | +0·57 |
| | ·3 | +0·79 | +0·28 | −0·12 | +0·76 | +0·96 | +1·16 |
| | ·2 | +1·17 | +0·52 | +0·01 | +0·94 | +1·34 | +1·80 |
| | ·1 | +1·52 | +0·76 | +0·14 | +1·13 | +1·73 | +2·47 |
| | 0 | +1·87 | +0·99 | +0·26 | +1·33 | +2·14 | +3·18 |
| 36 | ·5 | −0·18 | −0·34 | −0·44 | +0·44 | +0·34 | +0·18 |
| | ·4 | +0·26 | −0·06 | −0·29 | +0·60 | +0·67 | +0·72 |
| | ·3 | +0·68 | +0·21 | −0·15 | +0·77 | +1·01 | +1·28 |
| | ·2 | +1·09 | +0·47 | −0·02 | +0·94 | +1·36 | +1·86 |
| | ·1 | +1·49 | +0·73 | +0·12 | +1·11 | +1·71 | +2·46 |
| | 0 | +1·88 | +0·97 | +0·25 | +1·29 | +2·08 | +3·08 |
| 144 | ·5 | −0·35 | −0·43 | −0·47 | +0·47 | +0·43 | +0·35 |
| | ·4 | +0·12 | −0·14 | −0·33 | +0·63 | +0·73 | +0·86 |
| | ·3 | +0·57 | +0·14 | −0·18 | +0·78 | +1·05 | +1·37 |
| | ·2 | +1·01 | +0·42 | −0·04 | +0·94 | +1·37 | +1·90 |
| | ·1 | +1·45 | +0·69 | +0·09 | +1·09 | +1·69 | +2·44 |
| | 0 | +1·88 | +0·96 | +0·23 | +1·25 | +2·01 | +2·98 |
| ∞ | ·5 | −0·50 | −0·50 | −0·50 | +0·50 | +0·50 | +0·50 |
| | ·4 | −0·02 | −0·21 | −0·36 | +0·64 | +0·79 | +0·98 |
| | ·3 | +0·45 | +0·08 | −0·21 | +0·79 | +1·08 | +1·45 |
| | ·2 | +0·93 | +0·37 | −0·07 | +0·93 | +1·37 | +1·93 |
| | ·1 | +1·40 | +0·66 | +0·07 | +1·07 | +1·66 | +2·40 |
| | 0 | +1·88 | +0·95 | +0·21 | +1·21 | +1·95 | +2·88 |

To obtain the limits of the probability of an event (observed to occur $a$ times out of $N$) corresponding to a given probability level $P$, divide the tabulated values by $N$, first interpolating if necessary by linear interpolation with reference to $p = a/N$. The limits of the expectation of Poisson distributions are given directly, taking $p = 0$. For $a > \frac{1}{2}N$ enter the table with $a' = N - a$. For $a > 10$ add the tabulated corrections to the limits of the expectation calculated from the standard error $\sqrt{\{a(1-p)\}}$, using asymptotic interpolation on $12/\sqrt{a}$.

# TABLE VIII2. DENSITIES OF ORGANISMS ESTIMATED BY THE DILUTION METHOD

(W. L. Stevens)

*Two-fold*

| | Number of Levels ($s$). | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 or more |
| 0·4 | ·757 | ·773 | ·781 | ·785 | ·787 | ·788 | ·789 | ·789 |
| 0·6 | ·622 | ·640 | ·649 | ·653 | ·655 | ·656 | ·657 | ·657 |
| 0·8 | ·537 | ·556 | ·566 | ·571 | ·573 | ·574 | ·575 | ·575 |
| 1·0 | ·479 | ·500 | ·511 | ·516 | ·518 | ·520 | ·520 | ·521 |
| 1·2 | ·437 | ·461 | ·472 | ·478 | ·480 | ·482 | ·482 | ·483 |
| 1·4 | ·406 | ·432 | ·444 | ·450 | ·453 | ·455 | ·456 | ·456 |
| 1·6 | ·381 | ·411 | ·424 | ·431 | ·435 | ·436 | ·437 | ·438 |
| 1·8 | ·361 | ·394 | ·410 | ·417 | ·421 | ·423 | ·424 | ·425 |
| 2·0 | ·344 | ·382 | ·399 | ·408 | ·412 | ·414 | ·415 | ·416 |
| 2·5 | | ·358 | ·382 | ·394 | ·399 | ·402 | ·403 | ·405 |
| 3·0 | | | ·370 | ·386 | ·394 | ·398 | ·400 | ·402 |
| 3·5 | | | | ·379 | ·390 | ·396 | ·399 | ·401 |
| 4·0 | | | | | ·386 | ·394 | ·397 | ·401 |
| 4·5 | | | | | | ·390 | ·396 | ·401 |
| 5·0 | | | | | | | ·394 | ·401 |
| | | | | | | | | ·401* |
| $y$ | | | | | | | | |
| 7·0 | | | | | | | | ·399 |
| 6·0 | | | | | | | | ·397 |
| 5·0 | | | | | | | ·394 | ·394 |
| 4·5 | | | | | | ·390 | ·390 | ·390 |
| 4·0 | | | | | ·386 | ·386 | ·386 | ·386 |
| 3·5 | | | | ·379 | ·379 | ·379 | ·379 | ·379 |
| 3·0 | | | ·370 | ·370 | ·370 | ·370 | ·370 | ·370 |
| 2·5 | | ·358 | ·356 | ·356 | ·356 | ·356 | ·356 | ·356 |
| 2·0 | ·344 | ·334 | ·334 | ·334 | ·334 | ·334 | ·334 | ·334 |
| 1·8 | ·327 | ·323 | ·323 | ·323 | ·323 | ·323 | ·323 | ·323 |
| 1·6 | ·311 | ·309 | ·309 | ·309 | ·309 | ·309 | ·309 | ·309 |
| 1·4 | ·293 | ·292 | ·292 | ·292 | ·292 | ·292 | ·292 | ·292 |
| 1·2 | ·271 | ·271 | ·271 | ·271 | ·271 | ·271 | ·271 | ·271 |
| 1·0 | ·245 | ·245 | ·245 | ·245 | ·245 | ·245 | ·245 | ·245 |
| 0·8 | ·212 | ·212 | ·212 | ·212 | ·212 | ·212 | ·212 | ·212 |
| 0·6 | ·167 | ·167 | ·167 | ·167 | ·167 | ·167 | ·167 | ·167 |
| 0·4 | ·101 | ·101 | ·101 | ·101 | ·101 | ·101 | ·101 | ·101 |

*Ten-fold (Three or more levels)*

| $x \leqslant 1$ | | $x>1, y>2$ | | | $y \leqslant 2$ | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | $K$ | $x$ | $K$ | $y$ | $K$ | $y$ | $K$ | $y$ |
| | | ·0 | ·763 | ·0 | ·761 | 2·0 | ·744 | 1·0 |
| | | ·1 | ·768 | ·9 | ·766 | 1·9 | ·744 | 0·9 |
| | | ·2 | ·768 | ·8 | ·764 | 1·8 | ·734 | 0·8 |
| | | ·3 | ·760 | ·7 | ·755 | 1·7 | ·712 | 0·7 |
| 0·4 | ·761 | ·4 | ·747 | ·6 | ·741 | 1·6 | ·684 | 0·6 |
| 0·5 | ·740 | ·5 | ·736 | ·5 | ·729 | 1·5 | ·658 | 0·5 |
| 0·6 | ·733 | ·6 | ·733 | ·4 | ·724 | 1·4 | ·638 | 0·4 |
| 0·7 | ·736 | ·7 | ·736 | ·3 | ·726 | 1·3 | | |
| 0·8 | ·744 | ·8 | ·744 | ·2 | ·732 | 1·2 | | |
| 0·9 | ·753 | ·9 | ·753 | ·1 | ·739 | 1·1 | | |
| 1·0 | ·763 | ·0 | ·763 | ·0 | ·744 | 1·0 | | |

When $x>1$ and $y>2$ enter the table with the decimal part of $x$ or $y$ only.

*Four-fold*

| | Number of Levels | | |
|---|---|---|---|
| $x$ | 4 | 5 | 6 or more |
| 0·4 | ·704 | ·706 | ·707 |
| 0·6 | ·615 | ·617 | ·618 |
| 0·8 | ·573 | ·576 | ·577 |
| 1·0 | ·555 | ·558 | ·559 |
| 1·5 | ·545 | ·551 | ·553 |
| 2·0 | ·537 | ·548 | ·551 |
| 2·5 | | ·545 | ·552 |
| | | | ·552* |
| $y$ | | | |
| 3·5 | | | ·550 |
| 3·0 | | | ·548 |
| 2·5 | | ·545 | ·545 |
| 2·0 | ·537 | ·537 | ·537 |
| 1·5 | ·522 | ·522 | ·522 |
| 1·0 | ·488 | ·488 | ·488 |
| 0·8 | ·464 | ·464 | ·464 |
| 0·6 | ·431 | ·431 | ·431 |
| 0·4 | ·375 | ·375 | ·375 |

Calculate the mean fertile level $x$, and the mean sterile level $y$, where $x$ is the number of fertile plates/number of cultures at each level ($n$), and $x+y$ is the number of levels. Enter the table with $x$ or $y$, as indicated, and determine the corresponding value of the tabular entry $K$. When $x$ and $y$ fall outside the tabulated range use the value marked with an asterisk (*). The estimate of the number, $\lambda$, of organisms in the quantity of the medium used for one culture at the highest concentration is then given by $\log \lambda = x \log a - K$, where $a$ is the dilution factor. The average value of the variance of the mean fertile level is $\frac{1}{n}\frac{\log 2}{\log a}$, and the average value of the variance of $\log \lambda$ is $\frac{1}{n}\log 2 \log a$ (see Introduction). Thus:

Two-fold

$$\log \lambda = 0{\cdot}30103\, x - K$$
$$\overline{V(x)} = 1/n$$
$$\overline{V(\log \lambda)} = 0{\cdot}091/n$$

Four-fold.

$$\log \lambda = 0{\cdot}60206\, x - K$$
$$\overline{V(x)} = 1/2n$$
$$\overline{V(\log \lambda)} = 0{\cdot}201/n$$

Ten-fold.

$$\log \lambda = x - K$$
$$\overline{V(x)} = 0{\cdot}301/n$$

## TABLE VIII3. SIGNIFICANCE OF LEADING PERIODIC COMPONENTS

| *n* | 5 per cent. | 1 per cent. | *n* | 5 per cent. | 1 per cent. | *n* | 5 per cent. | 1 per cent. |
|---|---|---|---|---|---|---|---|---|
| 5 | ·68377 | ·78874 | 20 | ·27040 | ·32971 | 35 | ·17513 | ·21338 |
| 6 | ·61615 | ·72179 | 21 | ·26060 | ·31783 | 36 | ·17124 | ·20860 |
| 7 | ·56115 | ·66440 | 22 | ·25155 | ·30683 | 37 | ·16754 | ·20405 |
| 8 | ·51569 | ·61517 | 23 | ·24315 | ·29661 | 38 | ·16400 | ·19970 |
| 9 | ·47749 | ·57271 | 24 | ·23534 | ·28709 | 39 | ·16062 | ·19554 |
| 10 | ·44495 | ·53584 | 25 | ·22805 | ·27819 | 40 | ·15738 | ·19156 |
| 11 | ·41688 | ·50357 | 26 | ·22123 | ·26986 | 41 | ·15429 | ·18776 |
| 12 | ·39240 | ·47510 | 27 | ·21483 | ·26205 | 42 | ·15132 | ·18411 |
| 13 | ·37085 | ·44982 | 28 | ·20883 | ·25470 | 43 | ·14847 | ·18060 |
| 14 | ·35172 | ·42722 | 29 | ·20317 | ·24778 | 44 | ·14573 | ·17724 |
| 15 | ·33461 | ·40689 | 30 | ·19784 | ·24124 | 45 | ·14310 | ·17401 |
| 16 | ·31922 | ·38851 | 31 | ·19280 | ·23506 | 46 | ·14057 | ·17089 |
| 17 | ·30529 | ·37180 | 32 | ·18803 | ·22921 | 47 | ·13814 | ·16789 |
| 18 | ·29262 | ·35655 | 33 | ·18351 | ·22366 | 48 | ·13579 | ·16501 |
| 19 | ·28104 | ·34257 | 34 | ·17921 | ·21839 | 49 | ·13353 | ·16222 |
| 20 | ·27040 | ·32971 | 35 | ·17513 | ·21338 | 50 | ·13135 | ·15954 |

*Table of g*; for testing the significance of the leading periodic component of $2n+1$ or $2n+2$ consecutive values. Each of $n$ periods contributes a certain fraction to the sum of squares for all $n$ periods, and $g$ is taken to be the largest of these fractions. If this exceeds the corresponding tabulated value, significant evidence of periodicity is indicated.

# Table VIII4. The Normal Probability Integral

| $x$ |  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·0 | 0· | 50000 | 49601 | 49202 | 48803 | 48405 | 48006 | 47608 | 47210 | 46812 | 46414 |
| 0·1 |  | 46017 | 45620 | 45224 | 44828 | 44433 | 44038 | 43644 | 43251 | 42858 | 42465 |
| 0·2 |  | 42074 | 41683 | 41294 | 40905 | 40517 | 40129 | 39743 | 39358 | 38974 | 38591 |
| 0·3 |  | 38209 | 37828 | 37448 | 37070 | 36693 | 36317 | 35942 | 35569 | 35197 | 34827 |
| 0·4 |  | 34458 | 34090 | 33724 | 33360 | 32997 | 32636 | 32276 | 31918 | 31561 | 31207 |
| 0·5 |  | 30854 | 30503 | 30153 | 29806 | 29460 | 29116 | 28774 | 28434 | 28096 | 27760 |
| 0·6 |  | 27425 | 27093 | 26763 | 26435 | 26109 | 25785 | 25463 | 25143 | 24825 | 24510 |
| 0·7 |  | 24196 | 23885 | 23576 | 23270 | 22965 | 22663 | 22363 | 22065 | 21770 | 21476 |
| 0·8 |  | 21186 | 20897 | 20611 | 20327 | 20045 | 19766 | 19489 | 19215 | 18943 | 18673 |
| 0·9 |  | 18406 | 18141 | 17879 | 17619 | 17361 | 17106 | 16853 | 16602 | 16354 | 16109 |
| 1·0 |  | 15866 | 15625 | 15386 | 15151 | 14917 | 14686 | 14457 | 14231 | 14007 | 13786 |
| 1·1 |  | 13567 | 13350 | 13136 | 12924 | 12714 | 12507 | 12302 | 12100 | 11900 | 11702 |
| 1·2 |  | 11507 | 11314 | 11123 | 10935 | 10749 | 10565 | 10383 | 10204 | 10027 | **98525** |
| 1·3 | 0·0 | 96800 | 95098 | 93418 | 91759 | 90123 | 88508 | 86915 | 85343 | 83793 | 82264 |
| 1·4 |  | 80757 | 79270 | 77804 | 76359 | 74934 | 73529 | 72145 | 70781 | 69437 | 68112 |
| 1·5 |  | 66807 | 65522 | 64255 | 63008 | 61780 | 60571 | 59380 | 58208 | 57053 | 55917 |
| 1·6 |  | 54799 | 53699 | 52616 | 51551 | 50503 | 49471 | 48457 | 47460 | 46479 | 45514 |
| 1·7 |  | 44565 | 43633 | 42716 | 41815 | 40930 | 40059 | 39204 | 38364 | 37538 | 36727 |
| 1·8 |  | 35930 | 35148 | 34380 | 33625 | 32884 | 32157 | 31443 | 30742 | 30054 | 29379 |
| 1·9 |  | 28717 | 28067 | 27429 | 26803 | 26190 | 25588 | 24998 | 24419 | 23852 | 23295 |
| 2·0 |  | 22750 | 22216 | 21692 | 21178 | 20675 | 20182 | 19699 | 19226 | 18763 | 18309 |
| 2·1 |  | 17864 | 17429 | 17003 | 16586 | 16177 | 15778 | 15386 | 15003 | 14629 | 14262 |
| 2·2 |  | 13903 | 13553 | 13209 | 12874 | 12545 | 12224 | 11911 | 11604 | 11304 | 11011 |
| 2·3 |  | 10724 | 10444 | 10170 | **99031** | **96419** | **93867** | **91375** | **88940** | **86563** | **84242** |
| 2·4 | 0·0$^{2}$ | 81975 | 79763 | 77603 | 75494 | 73436 | 71428 | 69469 | 67557 | 65691 | 63872 |
| 2·5 |  | 62097 | 60366 | 58677 | 57031 | 55426 | 53861 | 52336 | 50849 | 49400 | 47988 |
| 2·6 |  | 46612 | 45271 | 43965 | 42692 | 41453 | 40246 | 39070 | 37926 | 36811 | 35726 |
| 2·7 |  | 34670 | 33642 | 32641 | 31667 | 30720 | 29798 | 28901 | 28028 | 27179 | 26354 |
| 2·8 |  | 25551 | 24771 | 24012 | 23274 | 22557 | 21860 | 21182 | 20524 | 19884 | 19262 |
| 2·9 |  | 18658 | 18071 | 17502 | 16948 | 16411 | 15889 | 15382 | 14890 | 14412 | 13949 |
| 3·0 |  | 13499 | 13062 | 12639 | 12228 | 11829 | 11442 | 11067 | 10703 | 10350 | 10008 |
| 3·1 | 0·0$^{3}$ | 96760 | 93544 | 90426 | 87403 | 84474 | 81635 | 78885 | 76219 | 73638 | 71136 |
| 3·2 |  | 68714 | 66367 | 64095 | 61895 | 59765 | 57703 | 55706 | 53774 | 51904 | 50094 |
| 3·3 |  | 48342 | 46648 | 45009 | 43423 | 41889 | 40406 | 38971 | 37584 | 36243 | 34946 |
| 3·4 |  | 33693 | 32481 | 31311 | 30179 | 29086 | 28029 | 27009 | 26023 | 25071 | 24151 |
| 3·5 |  | 23263 | 22405 | 21577 | 20778 | 20006 | 19262 | 18543 | 17849 | 17180 | 16534 |
| 3·6 |  | 15911 | 15310 | 14730 | 14171 | 13632 | 13112 | 12611 | 12128 | 11662 | 11213 |
| 3·7 |  | 10780 | 10363 | **99611** | **95740** | **92010** | **88417** | **84957** | **81624** | **78414** | **75324** |
| 3·8 | 0·0$^{4}$ | 72348 | 69483 | 66726 | 64072 | 61517 | 59059 | 56694 | 54418 | 52228 | 50122 |
| 3·9 |  | 48096 | 46148 | 44274 | 42473 | 40741 | 39076 | 37475 | 35936 | 34458 | 33037 |
| 4·0 |  | 31671 | 30359 | 29099 | 27888 | 26726 | 25609 | 24536 | 23507 | 22518 | 21569 |
| 4·1 |  | 20658 | 19783 | 18944 | 18138 | 17365 | 16624 | 15912 | 15230 | 14575 | 13948 |
| 4·2 |  | 13346 | 12769 | 12215 | 11685 | 11176 | 10689 | 10221 | **97736** | **93447** | **89337** |
| 4·3 | 0·0$^{5}$ | 85399 | 81627 | 78015 | 74555 | 71241 | 68069 | 65031 | 62123 | 59340 | 56675 |
| 4·4 |  | 54125 | 51685 | 49350 | 47117 | 44979 | 42935 | 40980 | 39110 | 37322 | 35612 |
| 4·5 |  | 33977 | 32414 | 30920 | 29492 | 28127 | 26823 | 25577 | 24386 | 23249 | 22162 |
| 4·6 |  | 21125 | 20133 | 19187 | 18283 | 17420 | 16597 | 15810 | 15060 | 14344 | 13660 |
| 4·7 |  | 13008 | 12386 | 11792 | 11226 | 10686 | 10171 | **96796** | **92113** | **87648** | **83391** |
| 4·8 | 0·0$^{6}$ | 79333 | 75465 | 71779 | 68267 | 64920 | 61731 | 58693 | 55799 | 53043 | 50418 |
| 4·9 |  | 47918 | 45538 | 43272 | 41115 | 39061 | 37107 | 35247 | 33476 | 31792 | 30190 |

# TABLE IX. PROBITS

## Transformation of the Sigmoid Dosage Mortality Curve to a Straight Line. (C. I. Bliss.)

| | 0·0 | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | ... | 1·9098 | 2·1218 | 2·2522 | 2·3479 | 2·4242 | 2·4879 | 2·5427 | 2·5911 | 2·6344 | | | | | |
| 1 | 2·6737 | 2·7096 | 2·7429 | 2·7738 | 2·8027 | 2·8299 | 2·8556 | 2·8799 | 2·9031 | 2·9251 | | | | | |
| 2 | 2·9463 | 2·9665 | 2·9859 | 3·0046 | 3·0226 | 3·0400 | 3·0569 | 3·0732 | 3·0890 | 3·1043 | For more detail, see values for 95-100. | | | | |
| 3 | 3·1192 | 3·1337 | 3·1478 | 3·1616 | 3·1750 | 3·1881 | 3·2009 | 3·2134 | 3·2256 | 3·2376 | | | | | |
| 4 | 3·2493 | 3·2608 | 3·2721 | 3·2831 | 3·2940 | 3·3046 | 3·3151 | 3·3253 | 3·3354 | 3·3454 | | | | | |
| 5 | 3·3551 | 3·3648 | 3·3742 | 3·3836 | 3·3928 | 3·4018 | 3·4107 | 3·4195 | 3·4282 | 3·4368 | 9 | 18 | 27 | 36 | 45 |
| 6 | 3·4452 | 3·4536 | 3·4618 | 3·4699 | 3·4780 | 3·4859 | 3·4937 | 3·5015 | 3·5091 | 3·5167 | 8 | 16 | 24 | 32 | 40 |
| 7 | 3·5242 | 3·5316 | 3·5389 | 3·5462 | 3·5534 | 3·5605 | 3·5675 | 3·5745 | 3·5813 | 3·5882 | 7 | 14 | 21 | 28 | 36 |
| 8 | 3·5949 | 3·6016 | 3·6083 | 3·6148 | 3·6213 | 3·6278 | 3·6342 | 3·6405 | 3·6468 | 3·6531 | 6 | 13 | 19 | 26 | 32 |
| 9 | 3·6592 | 3·6654 | 3·6715 | 3·6775 | 3·6835 | 3·6894 | 3·6953 | 3·7012 | 3·7070 | 3·7127 | 6 | 12 | 18 | 24 | 30 |
| 10 | 3·7184 | 3·7241 | 3·7298 | 3·7354 | 3·7409 | 3·7464 | 3·7519 | 3·7574 | 3·7628 | 3·7681 | 6 | 11 | 17 | 22 | 28 |
| 11 | 3·7735 | 3·7788 | 3·7840 | 3·7893 | 3·7945 | 3·7996 | 3·8048 | 3·8099 | 3·8150 | 3·8200 | 5 | 10 | 16 | 21 | 26 |
| 12 | 3·8250 | 3·8300 | 3·8350 | 3·8399 | 3·8448 | 3·8497 | 3·8545 | 3·8593 | 3·8641 | 3·8689 | 5 | 10 | 15 | 20 | 24 |
| 13 | 3·8736 | 3·8783 | 3·8830 | 3·8877 | 3·8923 | 3·8969 | 3·9015 | 3·9061 | 3·9107 | 3·9152 | 5 | 9 | 14 | 18 | 23 |
| 14 | 3·9197 | 3·9242 | 3·9286 | 3·9331 | 3·9375 | 3·9419 | 3·9463 | 3·9506 | 3·9550 | 3·9593 | 4 | 9 | 13 | 18 | 22 |
| 15 | 3·9636 | 3·9678 | 3·9721 | 3·9763 | 3·9806 | 3·9848 | 3·9890 | 3·9931 | 3·9973 | 4·0014 | 4 | 8 | 13 | 17 | 21 |
| 16 | 4·0055 | 4·0096 | 4·0137 | 4·0178 | 4·0218 | 4·0259 | 4·0299 | 4·0339 | 4·0379 | 4·0419 | 4 | 8 | 12 | 16 | 20 |
| 17 | 4·0458 | 4·0498 | 4·0537 | 4·0576 | 4·0615 | 4·0654 | 4·0693 | 4·0731 | 4·0770 | 4·0808 | 4 | 8 | 12 | 16 | 19 |
| 18 | 4·0846 | 4·0884 | 4·0922 | 4·0960 | 4·0998 | 4·1035 | 4·1073 | 4·1110 | 4·1147 | 4·1184 | 4 | 8 | 11 | 15 | 19 |
| 19 | 4·1221 | 4·1258 | 4·1295 | 4·1331 | 4·1367 | 4·1404 | 4·1440 | 4·1476 | 4·1512 | 4·1548 | 4 | 7 | 11 | 15 | 18 |
| 20 | 4·1584 | 4·1619 | 4·1655 | 4·1690 | 4·1726 | 4·1761 | 4·1796 | 4·1831 | 4·1866 | 4·1901 | 4 | 7 | 11 | 14 | 18 |
| 21 | 4·1936 | 4·1970 | 4·2005 | 4·2039 | 4·2074 | 4·2108 | 4·2142 | 4·2176 | 4·2210 | 4·2244 | 3 | 7 | 10 | 14 | 17 |
| 22 | 4·2278 | 4·2312 | 4·2345 | 4·2379 | 4·2412 | 4·2446 | 4·2479 | 4·2512 | 4·2546 | 4·2579 | 3 | 7 | 10 | 13 | 17 |
| 23 | 4·2612 | 4·2644 | 4·2677 | 4·2710 | 4·2743 | 4·2775 | 4·2808 | 4·2840 | 4·2872 | 4·2905 | 3 | 7 | 10 | 13 | 16 |
| 24 | 4·2937 | 4·2969 | 4·3001 | 4·3033 | 4·3065 | 4·3097 | 4·3129 | 4·3160 | 4·3192 | 4·3224 | 3 | 6 | 10 | 13 | 16 |
| 25 | 4·3255 | 4·3287 | 4·3318 | 4·3349 | 4·3380 | 4·3412 | 4·3443 | 4·3474 | 4·3505 | 4·3536 | 3 | 6 | 9 | 12 | 16 |
| 26 | 4·3567 | 4·3597 | 4·3628 | 4·3659 | 4·3689 | 4·3720 | 4·3750 | 4·3781 | 4·3811 | 4·3842 | 3 | 6 | 9 | 12 | 15 |
| 27 | 4·3872 | 4·3902 | 4·3932 | 4·3962 | 4·3992 | 4·4022 | 4·4052 | 4·4082 | 4·4112 | 4·4142 | 3 | 6 | 9 | 12 | 15 |
| 28 | 4·4172 | 4·4201 | 4·4231 | 4·4260 | 4·4290 | 4·4319 | 4·4349 | 4·4378 | 4·4408 | 4·4437 | 3 | 6 | 9 | 12 | 15 |
| 29 | 4·4466 | 4·4495 | 4·4524 | 4·4554 | 4·4583 | 4·4612 | 4·4641 | 4·4670 | 4·4698 | 4·4727 | 3 | 6 | 9 | 12 | 14 |
| 30 | 4·4756 | 4·4785 | 4·4813 | 4·4842 | 4·4871 | 4·4899 | 4·4928 | 4·4956 | 4·4985 | 4·5013 | 3 | 6 | 9 | 11 | 14 |
| 31 | 4·5041 | 4·5070 | 4·5098 | 4·5126 | 4·5155 | 4·5183 | 4·5211 | 4·5239 | 4·5267 | 4·5295 | 3 | 6 | 8 | 11 | 14 |
| 32 | 4·5323 | 4·5351 | 4·5379 | 4·5407 | 4·5435 | 4·5462 | 4·5490 | 4·5518 | 4·5546 | 4·5573 | 3 | 6 | 8 | 11 | 14 |
| 33 | 4·5601 | 4·5628 | 4·5656 | 4·5684 | 4·5711 | 4·5739 | 4·5766 | 4·5793 | 4·5821 | 4·5848 | 3 | 5 | 8 | 11 | 14 |
| 34 | 4·5875 | 4·5903 | 4·5930 | 4·5957 | 4·5984 | 4·6011 | 4·6039 | 4·6066 | 4·6093 | 4·6120 | 3 | 5 | 8 | 11 | 14 |
| 35 | 4·6147 | 4·6174 | 4·6201 | 4·6228 | 4·6255 | 4·6281 | 4·6308 | 4·6335 | 4·6362 | 4·6389 | 3 | 5 | 8 | 11 | 13 |
| 36 | 4·6415 | 4·6442 | 4·6469 | 4·6495 | 4·6522 | 4·6549 | 4·6575 | 4·6602 | 4·6628 | 4·6655 | 3 | 5 | 8 | 11 | 13 |
| 37 | 4·6681 | 4·6708 | 4·6734 | 4·6761 | 4·6787 | 4·6814 | 4·6840 | 4·6866 | 4·6893 | 4·6919 | 3 | 5 | 8 | 11 | 13 |
| 38 | 4·6945 | 4·6971 | 4·6998 | 4·7024 | 4·7050 | 4·7076 | 4·7102 | 4·7129 | 4·7155 | 4·7181 | 3 | 5 | 8 | 10 | 13 |
| 39 | 4·7207 | 4·7233 | 4·7259 | 4·7285 | 4·7311 | 4·7337 | 4·7363 | 4·7389 | 4·7415 | 4·7441 | 3 | 5 | 8 | 10 | 13 |
| 40 | 4·7467 | 4·7492 | 4·7518 | 4·7544 | 4·7570 | 4·7596 | 4·7622 | 4·7647 | 4·7673 | 4·7699 | 3 | 5 | 8 | 10 | 13 |
| 41 | 4·7725 | 4·7750 | 4·7776 | 4·7802 | 4·7827 | 4·7853 | 4·7879 | 4·7904 | 4·7930 | 4·7955 | 3 | 5 | 8 | 10 | 13 |
| 42 | 4·7981 | 4·8007 | 4·8032 | 4·8058 | 4·8083 | 4·8109 | 4·8134 | 4·8160 | 4·8185 | 4·8211 | 3 | 5 | 8 | 10 | 13 |
| 43 | 4·8236 | 4·8262 | 4·8287 | 4·8313 | 4·8338 | 4·8363 | 4·8389 | 4·8414 | 4·8440 | 4·8465 | 3 | 5 | 8 | 10 | 13 |
| 44 | 4·8490 | 4·8516 | 4·8541 | 4·8566 | 4·8592 | 4·8617 | 4·8642 | 4·8668 | 4·8693 | 4·8718 | 3 | 5 | 8 | 10 | 13 |
| 45 | 4·8743 | 4·8769 | 4·8794 | 4·8819 | 4·8844 | 4·8870 | 4·8895 | 4·8920 | 4·8945 | 4·8970 | 3 | 5 | 8 | 10 | 13 |
| 46 | 4·8996 | 4·9021 | 4·9046 | 4·9071 | 4·9096 | 4·9122 | 4·9147 | 4·9172 | 4·9197 | 4·9222 | 3 | 5 | 8 | 10 | 13 |
| 47 | 4·9247 | 4·9272 | 4·9298 | 4·9323 | 4·9348 | 4·9373 | 4·9398 | 4·9423 | 4·9448 | 4·9473 | 3 | 5 | 8 | 10 | 13 |
| 48 | 4·9498 | 4·9524 | 4·9549 | 4·9574 | 4·9599 | 4·9624 | 4·9649 | 4·9674 | 4·9699 | 4·9724 | 3 | 5 | 8 | 10 | 13 |
| 49 | 4·9749 | 4·9774 | 4·9799 | 4·9825 | 4·9850 | 4·9875 | 4·9900 | 4·9925 | 4·9950 | 4·9975 | 3 | 5 | 8 | 10 | 13 |

The probit corresponding to a given percentage is the normal deviate (increased by 5 to avoid negative values) exceeding this percentage in the population (see Table I).

TABLE IX. PROBITS—*continued*

|  | 0·0 | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 5·0000 | 5·0025 | 5·0050 | 5·0075 | 5·0100 | 5·0125 | 5·0150 | 5·0175 | 5·0201 | 5·0226 | 3 | 5 | 8 | 10 | 13 |
| 51 | 5·0251 | 5·0276 | 5·0301 | 5·0326 | 5·0351 | 5·0376 | 5·0401 | 5·0426 | 5·0451 | 5·0476 | 3 | 5 | 8 | 10 | 13 |
| 52 | 5·0502 | 5·0527 | 5·0552 | 5·0577 | 5·0602 | 5·0627 | 5·0652 | 5·0677 | 5·0702 | 5·0728 | 3 | 5 | 8 | 10 | 13 |
| 53 | 5·0753 | 5·0778 | 5·0803 | 5·0828 | 5·0853 | 5·0878 | 5·0904 | 5·0929 | 5·0954 | 5·0979 | 3 | 5 | 8 | 10 | 13 |
| 54 | 5·1004 | 5·1030 | 5·1055 | 5·1080 | 5·1105 | 5·1130 | 5·1156 | 5·1181 | 5·1206 | 5·1231 | 3 | 5 | 8 | 10 | 13 |
| 55 | 5·1257 | 5·1282 | 5·1307 | 5·1332 | 5·1358 | 5·1383 | 5·1408 | 5·1434 | 5·1459 | 5·1484 | 3 | 5 | 8 | 10 | 13 |
| 56 | 5·1510 | 5·1535 | 5·1560 | 5·1586 | 5·1611 | 5·1637 | 5·1662 | 5·1687 | 5·1713 | 5·1738 | 3 | 5 | 8 | 10 | 13 |
| 57 | 5·1764 | 5·1789 | 5·1815 | 5·1840 | 5·1866 | 5·1891 | 5·1917 | 5·1942 | 5·1968 | 5·1993 | 3 | 5 | 8 | 10 | 13 |
| 58 | 5·2019 | 5·2045 | 5·2070 | 5·2096 | 5·2121 | 5·2147 | 5·2173 | 5·2198 | 5·2224 | 5·2250 | 3 | 5 | 8 | 10 | 13 |
| 59 | 5·2275 | 5·2301 | 5·2327 | 5·2353 | 5·2378 | 5·2404 | 5·2430 | 5·2456 | 5·2482 | 5·2508 | 3 | 5 | 8 | 10 | 13 |
| 60 | 5·2533 | 5·2559 | 5·2585 | 5·2611 | 5·2637 | 5·2663 | 5·2689 | 5·2715 | 5·2741 | 5·2767 | 3 | 5 | 8 | 10 | 13 |
| 61 | 5·2793 | 5·2819 | 5·2845 | 5·2871 | 5·2898 | 5·2924 | 5·2950 | 5·2976 | 5·3002 | 5·3029 | 3 | 5 | 8 | 10 | 13 |
| 62 | 5·3055 | 5·3081 | 5·3107 | 5·3134 | 5·3160 | 5·3186 | 5·3213 | 5·3239 | 5·3266 | 5·3292 | 3 | 5 | 8 | 11 | 13 |
| 63 | 5·3319 | 5·3345 | 5·3372 | 5·3398 | 5·3425 | 5·3451 | 5·3478 | 5·3505 | 5·3531 | 5·3558 | 3 | 5 | 8 | 11 | 13 |
| 64 | 5·3585 | 5·3611 | 5·3638 | 5·3665 | 5·3692 | 5·3719 | 5·3745 | 5·3772 | 5·3799 | 5·3826 | 3 | 5 | 8 | 11 | 13 |
| 65 | 5·3853 | 5·3880 | 5·3907 | 5·3934 | 5·3961 | 5·3989 | 5·4016 | 5·4043 | 5·4070 | 5·4097 | 3 | 5 | 8 | 11 | 14 |
| 66 | 5·4125 | 5·4152 | 5·4179 | 5·4207 | 5·4234 | 5·4261 | 5·4289 | 5·4316 | 5·4344 | 5·4372 | 3 | 5 | 8 | 11 | 14 |
| 67 | 5·4399 | 5·4427 | 5·4454 | 5·4482 | 5·4510 | 5·4538 | 5·4565 | 5·4593 | 5·4621 | 5·4649 | 3 | 6 | 8 | 11 | 14 |
| 68 | 5·4677 | 5·4705 | 5·4733 | 5·4761 | 5·4789 | 5·4817 | 5·4845 | 5·4874 | 5·4902 | 5·4930 | 3 | 6 | 8 | 11 | 14 |
| 69 | 5·4959 | 5·4987 | 5·5015 | 5·5044 | 5·5072 | 5·5101 | 5·5129 | 5·5158 | 5·5187 | 5·5215 | 3 | 6 | 9 | 11 | 14 |
| 70 | 5·5244 | 5·5273 | 5·5302 | 5·5330 | 5·5359 | 5·5388 | 5·5417 | 5·5446 | 5·5476 | 5·5505 | 3 | 6 | 9 | 12 | 14 |
| 71 | 5·5534 | 5·5563 | 5·5592 | 5·5622 | 5·5651 | 5·5681 | 5·5710 | 5·5740 | 5·5769 | 5·5799 | 3 | 6 | 9 | 12 | 15 |
| 72 | 5·5828 | 5·5858 | 5·5888 | 5·5918 | 5·5948 | 5·5978 | 5·6008 | 5·6038 | 5·6068 | 5·6098 | 3 | 6 | 9 | 12 | 15 |
| 73 | 5·6128 | 5·6158 | 5·6189 | 5·6219 | 5·6250 | 5·6280 | 5·6311 | 5·6341 | 5·6372 | 5·6403 | 3 | 6 | 9 | 12 | 15 |
| 74 | 5·6433 | 5·6464 | 5·6495 | 5·6526 | 5·6557 | 5·6588 | 5·6620 | 5·6651 | 5·6682 | 5·6713 | 3 | 6 | 9 | 12 | 16 |
| 75 | 5·6745 | 5·6776 | 5·6808 | 5·6840 | 5·6871 | 5·6903 | 5·6935 | 5·6967 | 5·6999 | 5·7031 | 3 | 6 | 10 | 13 | 16 |
| 76 | 5·7063 | 5·7095 | 5·7128 | 5·7160 | 5·7192 | 5·7225 | 5·7257 | 5·7290 | 5·7323 | 5·7356 | 3 | 7 | 10 | 13 | 16 |
| 77 | 5·7388 | 5·7421 | 5·7454 | 5·7488 | 5·7521 | 5·7554 | 5·7588 | 5·7621 | 5·7655 | 5·7688 | 3 | 7 | 10 | 13 | 17 |
| 78 | 5·7722 | 5·7756 | 5·7790 | 5·7824 | 5·7858 | 5·7892 | 5·7926 | 5·7961 | 5·7995 | 5·8030 | 3 | 7 | 10 | 14 | 17 |
| 79 | 5·8064 | 5·8099 | 5·8134 | 5·8169 | 5·8204 | 5·8239 | 5·8274 | 5·8310 | 5·8345 | 5·8381 | 4 | 7 | 11 | 14 | 18 |
| 80 | 5·8416 | 5·8452 | 5·8488 | 5·8524 | 5·8560 | 5·8596 | 5·8633 | 5·8669 | 5·8705 | 5·8742 | 4 | 7 | 11 | 14 | 18 |
| 81 | 5·8779 | 5·8816 | 5·8853 | 5·8890 | 5·8927 | 5·8965 | 5·9002 | 5·9040 | 5·9078 | 5·9116 | 4 | 7 | 11 | 15 | 19 |
| 82 | 5·9154 | 5·9192 | 5·9230 | 5·9269 | 5·9307 | 5·9346 | 5·9385 | 5·9424 | 5·9463 | 5·9502 | 4 | 8 | 12 | 15 | 19 |
| 83 | 5·9542 | 5·9581 | 5·9621 | 5·9661 | 5·9701 | 5·9741 | 5·9782 | 5·9822 | 5·9863 | 5·9904 | 4 | 8 | 12 | 16 | 20 |
| 84 | 5·9945 | 5·9986 | 6·0027 | 6·0069 | 6·0110 | 6·0152 | 6·0194 | 6·0237 | 6·0279 | 6·0322 | 4 | 8 | 13 | 17 | 21 |
| 85 | 6·0364 | 6·0407 | 6·0450 | 6·0494 | 6·0537 | 6·0581 | 6·0625 | 6·0669 | 6·0714 | 6·0758 | 4 | 9 | 13 | 18 | 22 |
| 86 | 6·0803 | 6·0848 | 6·0893 | 6·0939 | 6·0985 | 6·1031 | 6·1077 | 6·1123 | 6·1170 | 6·1217 | 5 | 9 | 14 | 18 | 23 |
| 87 | 6·1264 | 6·1311 | 6·1359 | 6·1407 | 6·1455 | 6·1503 | 6·1552 | 6·1601 | 6·1650 | 6·1700 | 5 | 10 | 15 | 19 | 24 |
| 88 | 6·1750 | 6·1800 | 6·1850 | 6·1901 | 6·1952 | 6·2004 | 6·2055 | 6·2107 | 6·2160 | 6·2212 | 5 | 10 | 15 | 21 | 26 |
| 89 | 6·2265 | 6·2319 | 6·2372 | 6·2426 | 6·2481 | 6·2536 | 6·2591 | 6·2646 | 6·2702 | 6·2759 | 5 | 11 | 16 | 22 | 27 |
| 90 | 6·2816 | 6·2873 | 6·2930 | 6·2988 | 6·3047 | 6·3106 | 6·3165 | 6·3225 | 6·3285 | 6·3346 | 6 | 12 | 18 | 24 | 29 |
| 91 | 6·3408 | 6·3469 | 6·3532 | 6·3595 | 6·3658 | 6·3722 | 6·3787 | 6·3852 | 6·3917 | 6·3984 | 6 | 13 | 19 | 26 | 32 |
| 92 | 6·4051 | 6·4118 | 6·4187 | 6·4255 | 6·4325 | 6·4395 | 6·4466 | 6·4538 | 6·4611 | 6·4684 | 7 | 14 | 21 | 28 | 35 |
| 93 | 6·4758 | 6·4833 | 6·4909 | 6·4985 | 6·5063 | 6·5141 | 6·5220 | 6·5301 | 6·5382 | 6·5464 | 8 | 16 | 24 | 31 | 39 |
| 94 | 6·5548 | 6·5632 | 6·5718 | 6·5805 | 6·5893 | 6·5982 | 6·6072 | 6·6164 | 6·6258 | 6·6352 | 9 | 18 | 27 | 36 | 45 |
| 95 | 6·6449<br>97 | 6·6546<br>100 | 6·6646<br>101 | 6·6747<br>102 | 6·6849<br>105 | 6·6954<br>106 | 6·7060<br>109 | 6·7169<br>110 | 6·7279<br>113 | 6·7392<br>115 |  |  |  |  |  |
| 96 | 6·7507<br>117 | 6·7624<br>120 | 6·7744<br>122 | 6·7866<br>125 | 6·7991<br>128 | 6·8119<br>131 | 6·8250<br>134 | 6·8384<br>138 | 6·8522<br>141 | 6·8663<br>145 |  |  |  |  |  |
| 97 | 6·8808<br>149 | 6·8957<br>153 | 6·9110<br>158 | 6·9268<br>163 | 6·9431<br>169 | 6·9600<br>174 | 6·9774<br>180 | 6·9954<br>187 | 7·0141<br>194 | 7·0335<br>202 |  |  |  |  |  |

Continued on next page.

## TABLE IX. PROBITS—*continued*

| | 0·00 | 0·01 | 0·02 | 0·03 | 0·04 | 0·05 | 0·06 | 0·07 | 0·08 | 0·09 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 98·0 | 7·0537 | 7·0558 | 7·0579 | 7·0600 | 7·0621 | 7·0642 | 7·0663 | 7·0684 | 7·0706 | 7·0727 | 2 | 4 | 6 | 8 | 11 |
| 98·1 | 7·0749 | 7·0770 | 7·0792 | 7·0814 | 7·0836 | 7·0858 | 7·0880 | 7·0902 | 7·0924 | 7·0947 | 2 | 4 | 7 | 9 | 11 |
| 98·2 | 7·0969 | 7·0992 | 7·1015 | 7·1038 | 7·1061 | 7·1084 | 7·1107 | 7·1130 | 7·1154 | 7·1177 | 2 | 5 | 7 | 9 | 12 |
| 98·3 | 7·1201 | 7·1224 | 7·1248 | 7·1272 | 7·1297 | 7·1321 | 7·1345 | 7·1370 | 7·1394 | 7·1419 | 2 | 5 | 7 | 10 | 12 |
| 98·4 | 7·1444 | 7·1469 | 7·1494 | 7·1520 | 7·1545 | 7·1571 | 7·1596 | 7·1622 | 7·1648 | 7·1675 | 3 | 5 | 8 | 10 | 13 |
| 98·5 | 7·1701 | 7·1727 | 7·1754 | 7·1781 | 7·1808 | 7·1835 | 7·1862 | 7·1890 | 7·1917 | 7·1945 | 3 | 5 | 8 | 11 | 14 |
| 98·6 | 7·1973 | 7·2001 | 7·2029 | 7·2058 | 7·2086 | 7·2115 | 7·2144 | 7·2173 | 7·2203 | 7·2232 | 3 | 6 | 9 | 12 | 14 |
| 98·7 | 7·2262 | 7·2292 | 7·2322 | 7·2353 | 7·2383 | 7·2414 | 7·2445 | 7·2476 | 7·2508 | 7·2539 | 3 | 6 | 9 | 12 | 15 |
| 98·8 | 7·2571 | 7·2603 | 7·2636 | 7·2668 | 7·2701 | 7·2734 | 7·2768 | 7·2801 | 7·2835 | 7·2869 | 3 | 7 | 10 | 13 | 17 |
| 98·9 | 7·2904 | 7·2938 | 7·2973 | 7·3009 | 7·3044 | 7·3080 | 7·3116 | 7·3152 | 7·3189 | 7·3226 | 4 | 7 | 11 | 14 | 18 |
| 99·0 | 7·3263 | 7·3301 | 7·3339 | 7·3378 | 7·3416 | 7·3455 | 7·3495 | 7·3535 | 7·3575 | 7·3615 | 4 | 8 | 12 | 16 | 20 |
| 99·1 | 7·3656 | 7·3698 | 7·3739 | 7·3781 | 7·3824 | 7·3867 | 7·3911 | 7·3954 | 7·3999 | 7·4044 | 4 | 9 | 13 | 17 | 22 |
| 99·2 | 7·4089 | 7·4135 | 7·4181 | 7·4228 | 7·4276 | 7·4324 | 7·4372 | 7·4422 | 7·4471 | 7·4522 | 5 | 10 | 14 | 19 | 24 |
| 99·3 | 7·4573 | 7·4624 | 7·4677 | 7·4730 | 7·4783 | 7·4838 | 7·4893 | 7·4949 | 7·5006 | 7·5063 | 5 | 11 | 16 | 22 | 27 |
| 99·4 | 7·5121 | 7·5181 | 7·5241 | 7·5302 | 7·5364 | 7·5427 | 7·5491 | 7·5556 | 7·5622 | 7·5690 | 6 | 13 | 19 | 25 | 32 |
| 99·5 | 7·5758 | 7·5828 | 7·5899 | 7·5972 | 7·6045 | 7·6121 | 7·6197 | 7·6276 | 7·6356 | 7·6437 | | | | | |
| 99·6 | 7·6521 | 7·6606 | 7·6693 | 7·6783 | 7·6874 | 7·6968 | 7·7065 | 7·7164 | 7·7266 | 7·7370 | | | | | |
| 99·7 | 7·7478 | 7·7589 | 7·7703 | 7·7822 | 7·7944 | 7·8070 | 7·8202 | 7·8338 | 7·8480 | 7·8627 | | | | | |
| 99·8 | 7·8782 | 7·8943 | 7·9112 | 7·9290 | 7·9478 | 7·9677 | 7·9889 | 8·0115 | 8·0357 | 8·0618 | | | | | |
| 99·9 | 8·0902 | 8·1214 | 8·1559 | 8·1947 | 8·2389 | 8·2905 | 8·3528 | 8·4316 | 8·5401 | 8·7190 | | | | | |

## TABLE X. SIMPLE QUANTILES OF THE NORMAL DISTRIBUTION

Proper Fractions, with Denominators up to 30, and the corresponding Normal Deviates

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *1/30 | 1·8339 | *1/10 | 1·2816 | 5/27 | 0·8958 | 8/29 | 0·5952 | *11/30 | 0·3407 | 9/20 | 0·1257 |
| 1/29 | 1·8187 | 3/29 | 1·2621 | 3/16 | 0·8871 | 5/18 | 0·5895 | 7/19 | 0·3360 | 5/11 | 0·1142 |
| 1/28 | 1·8028 | 2/19 | 1·2521 | 4/21 | 0·8761 | 7/25 | 0·5828 | 10/27 | 0·3309 | 11/24 | 0·1046 |
| 1/27 | 1·7861 | 3/28 | 1·2419 | 5/26 | 0·8694 | 2/7 | 0·5661 | 3/8 | 0·3186 | 6/13 | 0·0966 |
| 1/26 | 1·7688 | 1/9 | 1·2206 | *1/5 | 0·8416 | 7/24 | 0·5485 | 11/29 | 0·3073 | 13/28 | 0·0896 |
| 1/25 | 1·7507 | 3/26 | 1·1984 | 6/29 | 0·8172 | 5/17 | 0·5414 | 8/21 | 0·3030 | *7/15 | 0·0836 |
| 1/24 | 1·7317 | 2/17 | 1·1868 | 5/24 | 0·8122 | 8/27 | 0·5351 | 5/13 | 0·2934 | 8/17 | 0·0738 |
| 1/23 | 1·7117 | 3/25 | 1·1750 | 4/19 | 0·8046 | *3/10 | 0·5244 | 7/18 | 0·2822 | 9/19 | 0·0660 |
| 1/22 | 1·6906 | 1/8 | 1·1503 | 3/14 | 0·7916 | 7/23 | 0·5119 | 9/23 | 0·2759 | 10/21 | 0·0597 |
| 1/21 | 1·6684 | 3/23 | 1·1244 | 5/23 | 0·7810 | 4/13 | 0·5024 | 11/28 | 0·2719 | 11/23 | 0·0545 |
| 1/20 | 1·6449 | *2/15 | 1·1108 | 2/9 | 0·7647 | 9/29 | 0·4949 | *2/5 | 0·2533 | 12/25 | 0.0502 |
| 1/19 | 1·6199 | 3/22 | 1·0968 | 5/22 | 0·7479 | 5/16 | 0·4888 | 11/27 | 0·2342 | 13/27 | 0·0464 |
| 1/18 | 1·5932 | 4/29 | 1·0897 | 3/13 | 0·7363 | 6/19 | 0·4795 | 9/22 | 0·2299 | 14/29 | 0·0432 |
| 1/17 | 1·5648 | 1/7 | 1·0676 | *7/30 | 0·7279 | 7/22 | 0·4728 | 7/17 | 0·2230 | *1/2 | 0 |
| 1/16 | 1·5341 | 4/27 | 1·0444 | 4/17 | 0·7215 | 8/25 | 0·4677 | 12/29 | 0·2178 | | |
| *1/15 | 1·5011 | 3/20 | 1·0364 | 5/21 | 0·7124 | 9/28 | 0·4637 | 5/12 | 0·2104 | | |
| 2/29 | 1·4835 | 2/13 | 1·0201 | 6/25 | 0·7063 | *1/3 | 0·4307 | 8/19 | 0·1992 | | |
| 1/14 | 1·4652 | 3/19 | 1·0031 | 7/29 | 0·7019 | 10/29 | 0·3993 | 11/26 | 0·1940 | | |
| 2/27 | 1·4461 | 4/25 | 0·9945 | 1/4 | 0·6745 | 9/26 | 0·3957 | 3/7 | 0·1800 | | |
| 1/13 | 1·4261 | *1/6 | 0·9674 | 7/27 | 0·6456 | 8/23 | 0·3912 | *13/30 | 0·1679 | | |
| 2/25 | 1·4051 | 5/29 | 0·9447 | 6/23 | 0·6407 | 7/20 | 0·3853 | 10/23 | 0·1642 | | |
| 1/12 | 1·3829 | 4/23 | 0·9388 | 5/19 | 0·6336 | 6/17 | 0·3774 | 7/16 | 0·1573 | | |
| 2/23 | 1·3597 | 3/17 | 0·9289 | *4/15 | 0·6229 | 5/14 | 0·3661 | 11/25 | 0·1510 | | |
| 1/11 | 1·3352 | 5/28 | 0·9208 | 7/26 | 0·6151 | 9/25 | 0·3585 | 4/9 | 0·1397 | | |
| 2/21 | 1·3092 | 2/11 | 0·9085 | 3/11 | 0·6046 | 4/11 | 0·3488 | 13/29 | 0·1300 | | |

Many fractions with higher denominators may be quickly obtained by simple interpolation between adjacent values, *e.g.* from 5/13 and 7/18, because $\frac{5+7}{13+18}=\frac{12}{31}$, the deviate for 12/31 is judged to be $0{\cdot}2822+\frac{13}{31}({\cdot}0112)=0{\cdot}2868{,}97$. True value 0·2868,94. Each thirtieth is marked by an asterisk.

To convert to probit values subtract from 5, and for fractions $\lambda$ greater than ½ add the value for $1-\lambda$ to 5.

# TABLE XI. PROBITS

Weighting Coefficients and Probit Values to be used for Final Adjustments
(Adapted from Bliss, 1935)

| Expected Probit $Y$. | Minimum Working Probit $Y-P/Z$. | Range $1/Z$ | Maximum working Probit $Y+Q/Z$. | Weighting Coefficient $Z^2/PQ$. | Expected Probit $Y$. | Minimum Working Probit $Y-P/Z$. | Range $1/Z$ | Maximum working Probit $Y+Q/Z$. | Weighting Coefficient $Z^2/PQ$. |
|---|---|---|---|---|---|---|---|---|---|
| 1·1 | 0·8579 | 5034 | 5035 | ·00082 | 5·0 | 3·7467 | 2·5066 | 6·2533 | ·63662 |
| 1·2 | 0·9522 | 3425 | 3426 | ·00118 | 5·1 | 3·7401 | 2·5192 | 6·2593 | ·63431 |
| 1·3 | 1·0462 | 2354 | 2355 | ·00167 | 5·2 | 3·7186 | 2·5573 | 6·2759 | ·62742 |
| 1·4 | 1·1400 | 1634 | 1635 | ·00235 | 5·3 | 3·6798 | 2·6220 | 6·3018 | ·61609 |
| 1·5 | 1·2335 | 1146 | 1147 | ·00327 | 5·4 | 3·6203 | 2·7154 | 6·3357 | ·60052 |
| 1·6 | 1·3266 | 811·5 | 812·8 | ·00451 | 5·5 | 3·5360 | 2·8404 | 6·3764 | ·58099 |
| 1·7 | 1·4194 | 580·5 | 581·9 | ·00614 | 5·6 | 3·4220 | 3·0010 | 6·4230 | ·55788 |
| 1·8 | 1·5118 | 419·4 | 420·9 | ·00828 | 5·7 | 3·2724 | 3·2025 | 6·4749 | ·53159 |
| 1·9 | 1·6038 | 306·1 | 307·7 | ·01104 | 5·8 | 3·0794 | 3·4519 | 6·5313 | ·50260 |
| 2·0 | 1·6954 | 225·6 | 227·3 | ·01457 | 5·9 | 2·8335 | 3·7582 | 6·5917 | ·47144 |
| 2·1 | 1·7866 | 168·00 | 169·79 | ·01903 | 6·0 | 2·5230 | 4·1327 | 6·6557 | ·43863 |
| 2·2 | 1·8772 | 126·34 | 128·22 | ·02459 | 6·1 | 2·1324 | 4·5903 | 6·7227 | ·40474 |
| 2·3 | 1·9673 | 95·96 | 97·93 | ·03143 | 6·2 | 1·6429 | 5·1497 | 6·7926 | ·37031 |
| 2·4 | 2·0568 | 73·62 | 75·68 | ·03977 | 6·3 | 1·0295 | 5·8354 | 6·8649 | ·33589 |
| 2·5 | 2·1457 | 57·05 | 59·20 | ·04979 | 6·4 | 0·2606 | 6·6788 | 6·9394 | ·30199 |
| 2·6 | 2·2340 | 44·654 | 46·888 | ·06169 | 6·5 | —0·705 | 7·721 | 7·0158 | ·26907 |
| 2·7 | 2·3214 | 35·302 | 37·623 | ·07563 | 6·6 | —1·921 | 9·015 | 7·0940 | ·23753 |
| 2·8 | 2·4081 | 28·189 | 30·597 | ·09179 | 6·7 | —3·459 | 10·633 | 7·1739 | ·20774 |
| 2·9 | 2·4938 | 22·736 | 25·230 | ·11026 | 6·8 | —5·411 | 12·666 | 7·2551 | ·17994 |
| 3·0 | 2·5786 | 18·522 | 21·101 | ·13112 | 6·9 | —7·902 | 15·240 | 7·3376 | ·15436 |
| 3·1 | 2·6624 | 15·240 | 17·902 | ·15436 | 7·0 | —11·101 | 18·522 | 7·4214 | ·13112 |
| 3·2 | 2·7449 | 12·666 | 15·411 | ·17994 | 7·1 | —15·230 | 22·736 | 7·5062 | ·11026 |
| 3·3 | 2·8261 | 10·633 | 13·459 | ·20774 | 7·2 | —20·597 | 28·189 | 7·5919 | ·09179 |
| 3·4 | 2·9060 | 9·015 | 11·921 | ·23753 | 7·3 | —27·623 | 35·302 | 7·6786 | ·07564 |
| 3·5 | 2·9842 | 7·721 | 10·705 | ·26907 | 7·4 | —36·888 | 44·654 | 7·7661 | ·06168 |
| 3·6 | 3·0606 | 6·6788 | 9·7394 | ·30199 | 7·5 | —49·20 | 57·05 | 7·8543 | ·04979 |
| 3·7 | 3·1351 | 5·8354 | 8·9705 | ·33589 | 7·6 | —65·68 | 73·62 | 7·9432 | ·03977 |
| 3·8 | 3·2074 | 5·1497 | 8·3571 | ·37031 | 7·7 | —87·93 | 95·96 | 8·0327 | ·03143 |
| 3·9 | 3·2773 | 4·5903 | 7·8676 | ·40474 | 7·8 | —118·22 | 126·34 | 8·1228 | ·02458 |
| 4·0 | 3·3443 | 4·1327 | 7·4770 | ·43863 | 7·9 | —159·79 | 168·00 | 8·2134 | ·01903 |
| 4·1 | 3·4083 | 3·7582 | 7·1665 | ·47144 | 8·0 | —217·3 | 225·6 | 8·3046 | ·01457 |
| 4·2 | 3·4687 | 3·4519 | 6·9206 | ·50260 | 8·1 | —297·7 | 306·1 | 8·3962 | ·01104 |
| 4·3 | 3·5251 | 3·2025 | 6·7276 | ·53159 | 8·2 | —410·9 | 419·4 | 8·4882 | ·00828 |
| 4·4 | 3·5770 | 3·0010 | 6·5780 | ·55788 | 8·3 | —571·9 | 580·5 | 8·5806 | ·00614 |
| 4·5 | 3·6236 | 2·8404 | 6·4640 | ·58099 | 8·4 | —802·8 | 811·5 | 8·6734 | ·00451 |
| 4·6 | 3·6643 | 2·7154 | 6·3797 | ·60052 | 8·5 | —1137 | 1146 | 8·7666 | ·00327 |
| 4·7 | 3·6982 | 2·6220 | 6·3202 | ·61609 | 8·6 | —1625 | 1634 | 8·8600 | ·00235 |
| 4·8 | 3·7241 | 2·5573 | 6·2814 | ·62741 | 8·7 | —2345 | 2354 | 8·9538 | ·00167 |
| 4·9 | 3·7407 | 2·5192 | 6·2599 | ·63431 | 8·8 | —3416 | 3425 | 9·0478 | ·00118 |
| 5·0 | 3·7467 | 2·5066 | 6·2533 | ·63662 | 8·9 | —5025 | 5034 | 9·1421 | ·00082 |

In toxicology the probit values used are found by adding 5 to a normal deviate. The weighting coefficient for any test $Z^2/PQ$ is derived from the probit $Y$ given by some provisional formula. The appropriate score for a test in which an observed proportion $p$ die and $q$ survive is $y_e$, where $y_e = (Y-P/Z)+p/Z = (Y+Q/Z)-q/Z = q(Y-P/Z)+p(Y+Q/Z)$.

## TABLE XII. PROBITS

### Weighting Coefficients for Use when there is a Natural Mortality

(D. J. Finney)

| *Y* | *Q/Z* | Natural Mortality, *K* | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | ·00 | ·01 | ·02 | ·04 | ·06 | ·08 | ·10 | ·15 | ·20 | ·25 | ·30 | ·35 | ·40 |
| 1·1 | 5034 | ·001 | ·000 | | | | | | | | | | | |
| 1·2 | 3425 | ·001 | ·000 | | | | | | | | | | | |
| 1·3 | 2354 | ·002 | ·000 | | | | | | | | | | | |
| 1·4 | 1634 | ·002 | ·000 | | | | | | | | | | | |
| 1·5 | 1146 | ·003 | ·000 | | | | | | | | | | | |
| 1·6 | 811·2 | ·005 | ·000 | | | | | | | | | | | |
| 1·7 | 580·2 | ·006 | ·000 | | | | | | | | | | | |
| 1·8 | 419·1 | ·008 | ·001 | ·000 | | | | | | | | | | |
| 1·9 | 305·8 | ·011 | ·001 | ·000 | | | | | | | | | | |
| 2·0 | 225·3 | ·015 | ·002 | ·001 | ·000 | ·000 | | | | | | | | |
| 2·1 | 167·69 | ·019 | ·003 | ·002 | ·001 | ·001 | ·000 | ·000 | | | | | | |
| 2·2 | 126·02 | ·025 | ·005 | ·003 | ·001 | ·001 | ·001 | ·001 | ·000 | | | | | |
| 2·3 | 95·63 | ·031 | ·008 | ·005 | ·002 | ·002 | ·001 | ·001 | ·001 | ·000 | ·000 | | | |
| 2·4 | 73·28 | ·040 | ·013 | ·007 | ·004 | ·003 | ·002 | ·002 | ·001 | ·001 | ·001 | ·000 | ·000 | |
| 2·5 | 56·70 | ·050 | ·019 | ·012 | ·006 | ·004 | ·003 | ·003 | ·002 | ·001 | ·001 | ·001 | ·001 | ·000 |
| 2·6 | 44·288 | ·062 | ·028 | ·018 | ·010 | ·007 | ·005 | ·004 | ·003 | ·002 | ·001 | ·001 | ·001 | ·001 |
| 2·7 | 34·923 | ·076 | ·039 | ·026 | ·015 | ·011 | ·008 | ·007 | ·004 | ·003 | ·002 | ·002 | ·001 | ·001 |
| 2·8 | 27·797 | ·092 | ·053 | ·037 | ·023 | ·016 | ·013 | ·010 | ·007 | ·005 | ·004 | ·003 | ·002 | ·002 |
| 2·9 | 22·330 | ·110 | ·070 | ·051 | ·033 | ·024 | ·019 | ·015 | ·010 | ·007 | ·006 | ·004 | ·004 | ·003 |
| 3·0 | 18·101 | ·131 | ·091 | ·069 | ·046 | ·034 | ·027 | ·022 | ·015 | ·011 | ·008 | ·007 | ·005 | ·004 |
| 3·1 | 14·802 | ·154 | ·114 | ·090 | ·063 | ·048 | ·038 | ·032 | ·022 | ·016 | ·012 | ·010 | ·008 | ·006 |
| 3·2 | 12·211 | ·180 | ·140 | ·115 | ·083 | ·065 | ·053 | ·044 | ·030 | ·023 | ·018 | ·014 | ·011 | ·009 |
| 3·3 | 10·159 | ·208 | ·169 | ·142 | ·107 | ·085 | ·070 | ·059 | ·042 | ·031 | ·024 | ·020 | ·016 | ·013 |
| 3·4 | 8·521 | ·238 | ·201 | ·173 | ·135 | ·110 | ·092 | ·078 | ·056 | ·043 | ·034 | ·027 | ·022 | ·018 |
| 3·5 | 7·205 | ·269 | ·234 | ·206 | ·166 | ·138 | ·117 | ·101 | ·074 | ·057 | ·045 | ·036 | ·030 | ·025 |
| 3·6 | 6·1394 | ·302 | ·268 | ·241 | ·199 | ·169 | ·145 | ·127 | ·095 | ·074 | ·059 | ·048 | ·039 | ·033 |
| 3·7 | 5·2705 | ·336 | ·304 | ·277 | ·235 | ·202 | ·177 | ·156 | ·119 | ·094 | ·076 | ·062 | ·051 | ·043 |
| 3·8 | 4·5571 | ·370 | ·340 | ·315 | ·272 | ·238 | ·211 | ·188 | ·146 | ·117 | ·095 | ·078 | ·065 | ·055 |
| 3·9 | 3·9676 | ·405 | ·377 | ·352 | ·310 | ·275 | ·247 | ·222 | ·176 | ·142 | ·117 | ·097 | ·081 | ·068 |
| 4·0 | 3·4770 | ·439 | ·412 | ·389 | ·347 | ·313 | ·283 | ·258 | ·208 | ·170 | ·141 | ·119 | ·100 | ·084 |
| 4·1 | 3·0665 | ·471 | ·447 | ·424 | ·384 | ·350 | ·320 | ·294 | ·241 | ·200 | ·168 | ·142 | ·120 | ·102 |
| 4·2 | 2·7206 | ·503 | ·480 | ·458 | ·420 | ·386 | ·356 | ·330 | ·274 | ·231 | ·195 | ·166 | ·142 | ·121 |
| 4·3 | 2·4276 | ·532 | ·510 | ·490 | ·454 | ·421 | ·391 | ·364 | ·307 | ·261 | ·224 | ·192 | ·165 | ·142 |
| 4·4 | 2·1780 | ·558 | ·538 | ·519 | ·484 | ·453 | ·424 | ·397 | ·339 | ·292 | ·252 | ·218 | ·188 | ·163 |
| 4·5 | 1·9640 | ·581 | ·563 | ·545 | ·512 | ·481 | ·453 | ·427 | ·370 | ·321 | ·279 | ·243 | ·212 | ·184 |
| 4·6 | 1·7797 | ·601 | ·583 | ·567 | ·536 | ·507 | ·480 | ·454 | ·397 | ·348 | ·305 | ·268 | ·234 | ·205 |
| 4·7 | 1·6202 | ·616 | ·600 | ·585 | ·556 | ·528 | ·502 | ·477 | ·421 | ·372 | ·329 | ·290 | ·256 | ·224 |
| 4·8 | 1·4814 | ·627 | ·613 | ·598 | ·571 | ·545 | ·520 | ·496 | ·442 | ·394 | ·350 | ·311 | ·275 | ·243 |
| 4·9 | 1·3599 | ·634 | ·621 | ·607 | ·582 | ·557 | ·534 | ·511 | ·458 | ·411 | ·368 | ·328 | ·292 | ·259 |
| 5·0 | 1·2533 | ·637 | ·624 | ·612 | ·588 | ·565 | ·542 | ·521 | ·471 | ·424 | ·382 | ·343 | ·307 | ·273 |

## TABLE XI1. PROBITS—*continued*

Weighting Coefficients for Use when there is Natural Mortality

| *Y* | *Q*/*Z* | Natural Mortality, $K$<br>·00 | ·01 | ·02 | ·04 | ·06 | ·08 | ·10 | ·15 | ·20 | ·25 | ·30 | ·35 | ·40 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5·0 | 1·2533 | ·637 | ·624 | ·612 | ·588 | ·565 | ·542 | ·521 | ·471 | ·424 | ·382 | ·343 | ·307 | ·273 |
| 5·1 | 1·1593 | ·634 | ·623 | ·611 | ·589 | ·567 | ·546 | ·526 | ·478 | ·434 | ·392 | ·354 | ·318 | ·284 |
| 5·2 | 1·0759 | ·627 | ·617 | ·606 | ·585 | ·565 | ·546 | ·526 | ·481 | ·438 | ·398 | ·361 | ·325 | ·292 |
| 5·3 | 1·0018 | ·616 | ·606 | ·596 | ·577 | ·558 | ·540 | ·522 | ·479 | ·439 | ·400 | ·364 | ·329 | ·296 |
| 5·4 | 0·9357 | ·601 | ·591 | ·582 | ·565 | ·547 | ·530 | ·513 | ·473 | ·435 | ·398 | ·363 | ·330 | ·298 |
| 5·5 | 0·8764 | ·581 | ·573 | ·564 | ·548 | ·532 | ·516 | ·501 | ·463 | ·427 | ·392 | ·359 | ·327 | ·296 |
| 5·6 | 0·8230 | ·558 | ·550 | ·543 | ·528 | ·513 | ·498 | ·484 | ·449 | ·415 | ·382 | ·351 | ·320 | ·291 |
| 5·7 | 0·7749 | ·532 | ·525 | ·518 | ·504 | ·490 | ·477 | ·464 | ·431 | ·400 | ·369 | ·340 | ·311 | ·283 |
| 5·8 | 0·7313 | ·503 | ·496 | ·490 | ·477 | ·465 | ·453 | ·440 | ·411 | ·382 | ·353 | ·326 | ·299 | ·272 |
| 5·9 | 0·6917 | ·471 | ·466 | ·460 | ·449 | ·437 | ·426 | ·415 | ·388 | ·361 | ·334 | ·309 | ·284 | ·259 |
| 6·0 | 0·6557 | ·439 | ·433 | ·428 | ·418 | ·408 | ·398 | ·387 | ·363 | ·338 | ·314 | ·291 | ·267 | ·245 |
| 6·1 | 0·6227 | ·405 | ·400 | ·395 | ·386 | ·377 | ·368 | ·359 | ·336 | ·314 | ·292 | ·271 | ·249 | ·228 |
| 6·2 | 0·5926 | ·370 | ·366 | ·362 | ·354 | ·345 | ·337 | ·329 | ·309 | ·289 | ·269 | ·249 | ·230 | ·211 |
| 6·3 | 0·5649 | ·336 | ·332 | ·328 | ·321 | ·314 | ·306 | ·299 | ·281 | ·263 | ·245 | ·228 | ·210 | ·193 |
| 6·4 | 0·5394 | ·302 | ·299 | ·295 | ·289 | ·282 | ·276 | ·269 | ·253 | ·237 | ·222 | ·206 | ·190 | ·175 |
| 6·5 | 0·5158 | ·269 | ·266 | ·263 | ·258 | ·252 | ·246 | ·240 | ·226 | ·212 | ·198 | ·184 | ·171 | ·157 |
| 6·6 | 0·4940 | ·238 | ·235 | ·233 | ·228 | ·223 | ·218 | ·213 | ·200 | ·188 | ·176 | ·163 | ·151 | ·139 |
| 6·7 | 0·4739 | ·208 | ·206 | ·203 | ·199 | ·195 | ·190 | ·186 | ·175 | ·165 | ·154 | ·143 | ·133 | ·122 |
| 6·8 | 0·4551 | ·180 | ·178 | ·176 | ·172 | ·169 | ·165 | ·161 | ·152 | ·143 | ·134 | ·125 | ·115 | ·106 |
| 6·9 | 0·4376 | ·154 | ·153 | ·151 | ·148 | ·145 | ·142 | ·139 | ·131 | ·123 | ·115 | ·107 | ·099 | ·092 |
| 7·0 | 0·4214 | ·131 | ·130 | ·128 | ·126 | ·123 | ·120 | ·118 | ·111 | ·104 | ·098 | ·091 | ·085 | ·078 |
| 7·1 | 0·4062 | ·110 | ·109 | ·108 | ·106 | ·104 | ·101 | ·099 | ·093 | ·088 | ·082 | ·077 | ·071 | ·066 |
| 7·2 | 0·3919 | ·092 | ·091 | ·090 | ·088 | ·086 | ·084 | ·082 | ·078 | ·073 | ·069 | ·064 | ·059 | ·055 |
| 7·3 | 0·3786 | ·076 | ·075 | ·074 | ·073 | ·071 | ·070 | ·068 | ·064 | ·060 | ·057 | ·053 | ·049 | ·045 |
| 7·4 | 0·3661 | ·062 | ·061 | ·060 | ·059 | ·058 | ·057 | ·055 | ·052 | ·049 | ·046 | ·043 | ·040 | ·037 |
| 7·5 | 0·3543 | ·050 | ·049 | ·049 | ·048 | ·047 | ·046 | ·045 | ·042 | ·040 | ·037 | ·035 | ·032 | ·030 |
| 7·6 | 0·3432 | ·040 | ·039 | ·039 | ·038 | ·037 | ·037 | ·036 | ·034 | ·032 | ·030 | ·028 | ·026 | ·024 |
| 7·7 | 0·3327 | ·031 | ·031 | ·031 | ·030 | ·030 | ·029 | ·028 | ·027 | ·025 | ·024 | ·022 | ·020 | ·019 |
| 7·8 | 0·3228 | ·025 | ·024 | ·024 | ·024 | ·023 | ·023 | ·022 | ·021 | ·020 | ·018 | ·017 | ·016 | ·015 |
| 7·9 | 0·3134 | ·019 | ·019 | ·019 | ·018 | ·018 | ·018 | ·017 | ·016 | ·015 | ·014 | ·013 | ·012 | ·011 |
| 8·0 | 0·3046 | ·015 | ·014 | ·014 | ·014 | ·014 | ·013 | ·013 | ·012 | ·012 | ·011 | ·010 | ·009 | ·009 |
| 8·1 | 0·2962 | ·011 | ·011 | ·011 | ·011 | ·010 | ·010 | ·010 | ·009 | ·009 | ·008 | ·008 | ·007 | ·007 |
| 8·2 | 0·2882 | ·008 | ·008 | ·008 | ·008 | ·008 | ·008 | ·007 | ·007 | ·007 | ·006 | ·006 | ·005 | ·005 |
| 8·3 | 0·2806 | ·006 | ·006 | ·006 | ·006 | ·006 | ·006 | ·006 | ·005 | ·005 | ·005 | ·004 | ·004 | ·004 |
| 8·4 | 0·2734 | ·005 | ·004 | ·004 | ·004 | ·004 | ·004 | ·004 | ·004 | ·004 | ·003 | ·003 | ·003 | ·003 |
| 8·5 | 0·2666 | ·003 | ·003 | ·003 | ·003 | ·003 | ·003 | ·003 | ·003 | ·003 | ·002 | ·002 | ·002 | ·002 |
| 8·6 | 0·2600 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·001 |
| 8·7 | 0·2538 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·002 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 |
| 8·8 | 0·2478 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 |
| 8·9 | 0·2421 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·001 | ·000 |

## ANGULAR TRANSFORMATION

### TABLE XII. Transformation of Percentages to Degrees

| $p$% | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 5·7 | 8·1 | 10·0 | 11·5 | 12·9 | 14·2 | 15·3 | 16·4 | 17·5 |
| 10 | 18·4 | 19·4 | 20·3 | 21·1 | 22·0 | 22·8 | 23·6 | 24·4 | 25·1 | 25·8 |
| 20 | 26·6 | 27·3 | 28·0 | 28·7 | 29·3 | 30·0 | 30·7 | 31·3 | 31·9 | 32·6 |
| 30 | 33·2 | 33·8 | 34·4 | 35·1 | 35·7 | 36·3 | 36·9 | 37·5 | 38·1 | 38·6 |
| 40 | 39·2 | 39·8 | 40·4 | 41·0 | 41·6 | 42·1 | 42·7 | 43·3 | 43·9 | 44·4 |
| 50 | 45·0 | 45·6 | 46·1 | 46·7 | 47·3 | 47·9 | 48·4 | 49·0 | 49·6 | 50·2 |
| 60 | 50·8 | 51·4 | 51·9 | 52·5 | 53·1 | 53·7 | 54·3 | 54·9 | 55·6 | 56·2 |
| 70 | 56·8 | 57·4 | 58·1 | 58·7 | 59·3 | 60·0 | 60·7 | 61·3 | 62·0 | 62·7 |
| 80 | 63·4 | 64·2 | 64·9 | 65·6 | 66·4 | 67·2 | 68·0 | 68·9 | 69·7 | 70·6 |
| 90 | 71·6 | 72·5 | 73·6 | 74·7 | 75·8 | 77·1 | 78·5 | 80·0 | 81·9 | 84·3 |

In connection with an observed fraction $p$, it is often advisable to use the angular value $\phi$, such that $p = \sin^2 \phi$. $\phi$ rises to 90° at 100 per cent. The amount of information supplied by $n$ observations about $\phi$, in circular measure, is $4n$, or in degrees, $\frac{4\pi^2 n}{180^2} = \frac{\pi^2 n}{8100}$. Consequently in large samples, the variance of $\phi$, as estimated from a sample, is $\frac{8100}{\pi^2 n} = \frac{820{\cdot}7}{n}$ independently of the true value of $\phi$.

For small angles use $57{\cdot}3\sqrt{p}$ in preference to interpolation.

### TABLE XIII. Transformation of Proper Fractions to Degrees

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *1/30 | 10·5 | *1/10 | 18·4 | 5/27 | 25·5 | 8/29 | 31·7 | *11/30 | 37·3 | 9/20 | 42·1 |
| 1/29 | 10·7 | 3/29 | 18·8 | 3/16 | 25·7 | 5/18 | 31·8 | 7/19 | 37·4 | 5/11 | 42·4 |
| 1/28 | 10·9 | 2/19 | 18·9 | 4/21 | 25·9 | 7/25 | 31·9 | 10/27 | 37·5 | 11/24 | 42·6 |
| 1/27 | 11·1 | 3/28 | 19·1 | 5/26 | 26·0 | 2/7 | 32·3 | 3/8 | 37·8 | 6/13 | 42·8 |
| 1/26 | 11·3 | 1/9 | 19·5 | *1/5 | 26·6 | 7/24 | 32·7 | 11/29 | 38·0 | 13/28 | 43·0 |
| 1/25 | 11·5 | 3/26 | 19·9 | 6/29 | 27·1 | 5/17 | 32·8 | 8/21 | 38·1 | *7/15 | 43·1 |
| 1/24 | 11·8 | 2/17 | 20·1 | 5/24 | 27·2 | 8/27 | 33·0 | 5/13 | 38·3 | 8/17 | 43·3 |
| 1/23 | 12·0 | 3/25 | 20·3 | 4/19 | 27·3 | *3/10 | 33·2 | 7/18 | 38·6 | 9/19 | 43·5 |
| 1/22 | 12·3 | 1/8 | 20·7 | 3/14 | 27·6 | 7/23 | 33·5 | 9/23 | 38·7 | 10/21 | 43·6 |
| 1/21 | 12·6 | 3/23 | 21·2 | 5/23 | 27·8 | 4/13 | 33·7 | 11/28 | 38·8 | 11/23 | 43·8 |
| 1/20 | 12·9 | *2/15 | 21·4 | 2/9 | 28·1 | 9/29 | 33·9 | *2/5 | 39·2 | 12/25 | 43·9 |
| 1/19 | 13·3 | 3/22 | 21·7 | 5/22 | 28·5 | 5/16 | 34·0 | 11/27 | 39·7 | 13/27 | 43·9 |
| 1/18 | 13·6 | 4/29 | 21·8 | 3/13 | 28·7 | 6/19 | 34·2 | 9/22 | 39·8 | 14/29 | 44·0 |
| 1/17 | 14·0 | 1/7 | 22·2 | *7/30 | 28·9 | 7/22 | 34·3 | 7/17 | 39·9 | *1/2 | 45·0 |
| 1/16 | 14·5 | 4/27 | 22·6 | 4/17 | 29·0 | 8/25 | 34·4 | 12/29 | 40·0 | | |
| *1/15 | 15·0 | 3/20 | 22·8 | 5/21 | 29·2 | 9/28 | 34·5 | 5/12 | 40·2 | | |
| 2/29 | 15·2 | 2/13 | 23·1 | 6/25 | 29·3 | *1/3 | 35·3 | 8/19 | 40·5 | | |
| 1/14 | 15·5 | 3/19 | 23·4 | 7/29 | 29·4 | 10/29 | 36·0 | 11/26 | 40·6 | | |
| 2/27 | 15·8 | 4/25 | 23·6 | 1/4 | 30·0 | 9/26 | 36·0 | 3/7 | 40·9 | | |
| 1/13 | 16·1 | *1/6 | 24·1 | 7/27 | 30·6 | 8/23 | 36·1 | *13/30 | 41·2 | | |
| 2/25 | 16·4 | 5/29 | 24·5 | 6/23 | 30·7 | 7/20 | 36·3 | 10/23 | 41·3 | | |
| 1/12 | 16·8 | 4/23 | 24·6 | 5/19 | 30·9 | 6/17 | 36·4 | 7/16 | 41·4 | | |
| 2/23 | 17·2 | 3/17 | 24·8 | *4/15 | 31·1 | 5/14 | 36·7 | 11/25 | 41·6 | | |
| 1/11 | 17·5 | 5/28 | 25·0 | 7/26 | 31·3 | 9/25 | 36·9 | 4/9 | 41·8 | | |
| 2/21 | 18·0 | 2/11 | 25·2 | 3/11 | 31·5 | 4/11 | 37·1 | 13/29 | 42·0 | | |

For fractions exceeding ½, subtract the fraction from 1, and the angle from 90°. Each thirtieth is marked by an asterisk; for interpolation see Table X.

## TABLE XIV. ANGULAR TRANSFORMATION

### Angular Values for Final Adjustments

| Expected angular value $\phi$. | Maximal working angle $\phi + \frac{1}{2}\cot\phi$. | Range cosec $2\phi$. | Expected angular value $\phi$. | Maximal working angle $\phi + \frac{1}{2}\cot\phi$. | Range cosec $2\phi$. | Expected angular value $\phi$. | Maximal working angle $\phi + \frac{1}{2}\cot\phi$. | Range cosec $2\phi$. |
|---|---|---|---|---|---|---|---|---|
| 45 | 73·6 | 57·3 | 60 | 76·5 | 66·2 | 75 | 82·7 | 114·6 |
| 46 | 73·7 | 57·3 | 61 | 76·9 | 67·6 | 76 | 83·1 | 122·0 |
| 47 | 73·7 | 57·4 | 62 | 77·2 | 69·1 | 77 | 83·6 | 130·7 |
| 48 | 73·8 | 57·6 | 63 | 77·6 | 70·8 | 78 | 84·1 | 140·9 |
| 49 | 73·9 | 57·9 | 64 | 78·0 | 72·7 | 79 | 84·6 | 152·9 |
| 50 | 74·0 | 58·3 | 65 | 78·4 | 74·8 | 80 | 85·1 | 167·5 |
| 51 | 74·2 | 58·6 | 66 | 78·8 | 77·1 | 81 | 85·5 | 185·4 |
| 52 | 74·4 | 59·0 | 67 | 79·2 | 79·7 | 82 | 86·0 | 207·9 |
| 53 | 74·6 | 59·6 | 68 | 79·6 | 82·5 | 83 | 86·5 | 236·8 |
| 54 | 74·8 | 60·2 | 69 | 80·0 | 85·6 | 84 | 87·0 | 275·6 |
| 55 | 75·1 | 61·0 | 70 | 80·4 | 89·1 | 85 | 87·5 | 330·0 |
| 56 | 75·3 | 61·8 | 71 | 80·9 | 93·1 | 86 | 88·0 | 411·7 |
| 57 | 75·6 | 62·7 | 72 | 81·3 | 97·5 | 87 | 88·5 | 548·1 |
| 58 | 75·9 | 63·7 | 73 | 81·8 | 102·5 | 88 | 89·0 | 821·4 |
| 59 | 76·2 | 64·9 | 74 | 82·2 | 108·1 | 89 | 89·5 | 1641·7 |

This table supplies for the angular transformation the same facilities as does Table XI for probits. The weighting coefficient, 1/820·7, is constant for all angular values, and is only needed in testing homogeneity. The adjusted score for *expected* angular value $\phi$, is $\phi_e = \phi + \frac{1}{2}\cot\phi - q \operatorname{cosec} 2\phi$ where $q$ is the observed proportion surviving.

For values less than 45° subtract from 90°; the range is then as tabled; its *lower* extremity is at 90° less the value in the table.

## TABLE XIVI. SCORES FOR LINKAGE DATA FROM INTERCROSSES

| Recombination fraction. *y* per cent. | COUPLING | | | REPULSION | | |
|---|---|---|---|---|---|---|
| | Double dominant + | Single recessives − | Double recessive + | Double dominant − | Single recessives + | Double recessive − |
| 1 | ·6644072 | 99·4974874 | 2·0202020 | ·0099995 | ·0200020 | 200·0000000 |
| 2 | ·6620727 | 49·4949495 | 2·0408163 | ·0199960 | ·0400160 | 100·0000000 |
| 3 | ·6596620 | 32·8257191 | 2·0618557 | ·0299865 | ·0600540 | 66·6666667 |
| 4 | ·6571742 | 24·4897959 | 2·0833333 | ·0399680 | ·0801282 | 50·0000000 |
| 5 | ·6546081 | 19·4871795 | 2·1052632 | ·0499376 | ·1002506 | 40·0000000 |
| 6 | ·6519628 | 16·1512027 | 2·1276596 | ·0598922 | ·1204336 | 33·3333333 |
| 7 | ·6492373 | 13·7675796 | 2·1505376 | ·0698289 | ·1406894 | 28·5714286 |
| 8 | ·6464306 | 11·9791667 | 2·1739130 | ·0797448 | ·1610306 | 25·0000000 |
| 9 | ·6435416 | 10·5875509 | 2·1978022 | ·0896370 | ·1814699 | 22·2222222 |
| 10 | ·6405694 | 9·4736842 | 2·2222222 | ·0995025 | ·2020202 | 20·0000000 |
| 11 | ·6375130 | 8·5618086 | 2·2471910 | ·1093385 | ·2226946 | 18·1818182 |
| 12 | ·6343714 | 7·8014184 | 2·2727273 | ·1191422 | ·2435065 | 16·6666667 |
| 13 | ·6311437 | 7·1575483 | 2·2988506 | ·1289107 | ·2644695 | 15·3846154 |
| 14 | ·6278289 | 6·6052227 | 2·3255814 | ·1386413 | ·2855977 | 14·2857143 |
| 15 | ·6244261 | 6·1261261 | 2·3529412 | ·1483313 | ·3069054 | 13·3333333 |
| 16 | ·6209344 | 5·7065217 | 2·3809524 | ·1579779 | ·3284072 | 12·5000000 |
| 17 | ·6173528 | 5·3359049 | 2·4096386 | ·1675785 | ·3501184 | 11·7647059 |
| 18 | ·6136806 | 5·0061050 | 2·4390244 | ·1771305 | ·3720546 | 11·1111111 |
| 19 | ·6099168 | 4·7106717 | 2·4691358 | ·1866313 | ·3942318 | 10·5263158 |
| 20 | ·6060606 | 4·4444444 | 2·5000000 | ·1960784 | ·4166667 | 10·0000000 |
| 21 | ·6021112 | 4·2032455 | 2·5316456 | ·2054694 | ·4393765 | 9·5238095 |
| 22 | ·5980678 | 3·9836568 | 2·5641026 | ·2148018 | ·4623792 | 9·0909091 |
| 23 | ·5939296 | 3·7828543 | 2·5974026 | ·2240733 | ·4856932 | 8·6956522 |
| 24 | ·5896958 | 3·5984848 | 2·6315789 | ·2332815 | ·5093379 | 8·3333333 |
| 25 | ·5853659 | 3·4285714 | 2·6666667 | ·2424242 | ·5333333 | 8·0000000 |
| 26 | ·5809389 | 3·2714412 | 2·7027027 | ·2514993 | ·5577006 | 7·6923077 |
| 27 | ·5764144 | 3·1256690 | 2·7397260 | ·2605046 | ·5824614 | 7·4074074 |
| 28 | ·5717916 | 2·9900332 | 2·7777778 | ·2694380 | ·6076389 | 7·1428571 |
| 29 | ·5670700 | 2·8634805 | 2·8169014 | ·2782976 | ·6332569 | 6·8965517 |
| 30 | ·5622490 | 2·7450980 | 2·8571429 | ·2870813 | ·6593407 | 6·6666667 |
| 31 | ·5573281 | 2·6340905 | 2·8985507 | ·2957874 | ·6859166 | 6·4516129 |
| 32 | ·5523067 | 2·5297619 | 2·9411765 | ·3044140 | ·7130125 | 6·2500000 |
| 33 | ·5471845 | 2·4315006 | 2·9850746 | ·3129594 | ·7406576 | 6·0606061 |
| 34 | ·5419609 | 2·3387668 | 3·0303030 | ·3214218 | ·7688829 | 5·8823529 |
| 35 | ·5366357 | 2·2510823 | 3·0769231 | ·3297998 | ·7977208 | 5·7142857 |
| 36 | ·5312085 | 2·1680217 | 3·1250000 | ·3380917 | ·8272059 | 5·5555556 |
| 37 | ·5256790 | 2·0892058 | 3·1746032 | ·3462960 | ·8573746 | 5·4054054 |
| 38 | ·5200470 | 2·0142950 | 3·2258065 | ·3544115 | ·8882655 | 5·2631579 |
| 39 | ·5143122 | 1·9429846 | 3·2786885 | ·3624367 | ·9199198 | 5·1282051 |
| 40 | ·5084746 | 1·8750000 | 3·3333333 | ·3703704 | ·9523810 | 5·0000000 |
| 41 | ·5025340 | 1·8100936 | 3·3898305 | ·3782113 | ·9856954 | 4·8780488 |
| 42 | ·4964903 | 1·7480410 | 3·4482759 | ·3859585 | 1·0199126 | 4·7619048 |
| 43 | ·4903437 | 1·6886387 | 3·5087719 | ·3936107 | 1·0550853 | 4·6511628 |
| 44 | ·4840941 | 1·6317016 | 3·5714286 | ·4011670 | 1·0912698 | 4·5454545 |
| 45 | ·4777416 | 1·5770609 | 3·6363636 | ·4086266 | 1·1285266 | 4·4444444 |
| 46 | ·4712864 | 1·5245624 | 3·7037037 | ·4159884 | 1·1669203 | 4·3478261 |
| 47 | ·4647288 | 1·4740648 | 3·7735849 | ·4232518 | 1·2065203 | 4·2553191 |
| 48 | ·4580691 | 1·4254386 | 3·8461538 | ·4304161 | 1·2474012 | 4·1666667 |
| 49 | ·4513075 | 1·3785647 | 3·9215686 | ·4374805 | 1·2896434 | 4·0816327 |
| 50 | ·4444444 | 1·3333333 | 4·0000000 | ·4444444 | 1·3333333 | 4·0000000 |

# TABLE XIV2. SEGMENTAL FUNCTIONS

| $x$ | $\alpha(x)$ | $\beta(x)$ | $\gamma(x)$ | $\delta(x)$ |
|---|---|---|---|---|
| ·00 | 1·0000 000 | ·0000 000 | ·0000 000 | ·0000 000 |
| ·02 | 1·0000 000 | ·0200 000 | ·0002 000 | ·0000 013 |
| ·04 | 1·0000 001 | ·0400 000 | ·0008 000 | ·0000 107 |
| ·06 | 1·0000 005 | ·0600 000 | ·0018 000 | ·0000 360 |
| ·08 | 1·0000 017 | ·0800 000 | ·0032 000 | ·0000 853 |
| ·10 | 1·0000 042 | ·1000 001 | ·0050 000 | ·0001 667 |
| ·12 | 1·0000 086 | ·1200 002 | ·0072 000 | ·0002 880 |
| ·14 | 1·0000 160 | ·1400 004 | ·0098 000 | ·0004 573 |
| ·16 | 1·0000 273 | ·1600 009 | ·0128 000 | ·0006 827 |
| ·18 | 1·0000 437 | ·1800 016 | ·0162 000 | ·0009 720 |
| ·20 | 1·0000 667 | ·2000 027 | ·0200 001 | ·0013 333 |
| ·22 | 1·0000 976 | ·2200 043 | ·0242 002 | ·0017 747 |
| ·24 | 1·0001 382 | ·2400 066 | ·0288 003 | ·0023 040 |
| ·26 | 1·0001 904 | ·2600 099 | ·0338 004 | ·0029 293 |
| ·28 | 1·0002 561 | ·2800 143 | ·0392 007 | ·0036 587 |
| ·30 | 1·0003 375 | ·3000 203 | ·0450 010 | ·0045 000 |
| ·32 | 1·0004 369 | ·3200 280 | ·0512 015 | ·0054 614 |
| ·34 | 1·0005 568 | ·3400 379 | ·0578 021 | ·0065 508 |
| ·36 | 1·0006 998 | ·3600 504 | ·0648 030 | ·0077 762 |
| ·38 | 1·0008 688 | ·3800 660 | ·0722 042 | ·0091 456 |
| ·40 | 1·0010 667 | ·4000 853 | ·0800 057 | ·0106 670 |
| ·42 | 1·0012 966 | ·4201 089 | ·0882 076 | ·0123 485 |
| ·44 | 1·0015 617 | ·4401 374 | ·0968 101 | ·0141 980 |
| ·46 | 1·0018 657 | ·4601 716 | ·1058 132 | ·0162 235 |
| ·48 | 1·0022 119 | ·4802 123 | ·1152 170 | ·0184 332 |
| ·50 | 1·0026 043 | ·5002 604 | ·1250 217 | ·0208 349 |
| ·52 | 1·0030 466 | ·5203 168 | ·1352 275 | ·0234 367 |
| ·54 | 1·0035 431 | ·5403 826 | ·1458 344 | ·0262 467 |
| ·56 | 1·0040 979 | ·5604 590 | ·1568 428 | ·0292 728 |
| ·58 | 1·0047 155 | ·5805 470 | ·1682 529 | ·0325 230 |
| ·60 | 1·0054 004 | ·6006 480 | ·1800 648 | ·0360 056 |
| ·62 | 1·0061 573 | ·6207 635 | ·1922 789 | ·0397 283 |
| ·64 | 1·0069 912 | ·6408 948 | ·2048 954 | ·0436 994 |
| ·66 | 1·0079 070 | ·6610 437 | ·2179 148 | ·0479 268 |
| ·68 | 1·0089 100 | ·6812 117 | ·2313 373 | ·0524 187 |
| ·70 | 1·0100 056 | ·7014 007 | ·2451 634 | ·0571 830 |
| ·72 | 1·0111 992 | ·7216 126 | ·2593 935 | ·0622 279 |
| ·74 | 1·0124 966 | ·7418 494 | ·2740 281 | ·0675 614 |
| ·76 | 1·0139 037 | ·7621 132 | ·2890 677 | ·0731 917 |
| ·78 | 1·0154 263 | ·7824 063 | ·3045 128 | ·0791 269 |
| ·80 | 1·0170 708 | ·8027 310 | ·3203 641 | ·0853 749 |
| ·82 | 1·0188 435 | ·8230 900 | ·3366 223 | ·0919 441 |
| ·84 | 1·0207 508 | ·8434 857 | ·3532 880 | ·0988 426 |
| ·86 | 1·0227 994 | ·8639 209 | ·3703 620 | ·1060 784 |
| ·88 | 1·0249 962 | ·8843 986 | ·3878 451 | ·1136 598 |
| ·90 | 1·0273 482 | ·9049 218 | ·4057 382 | ·1215 949 |
| ·92 | 1·0298 624 | ·9254 936 | ·4240 423 | ·1298 920 |
| ·94 | 1·0325 463 | ·9461 174 | ·4427 583 | ·1385 593 |
| ·96 | 1·0354 073 | ·9667 967 | ·4618 873 | ·1476 051 |
| ·98 | 1·0384 531 | ·9875 350 | ·4814 306 | ·1570 376 |
| 1·00 | 1·0416 915 | 1·0083 361 | ·5013 892 | ·1668 651 |

## TABLE XV. LATIN SQUARES

### The 4×4 Latin Squares

First Transformation Set:
3 Self-Conjugate Standard Squares

A B C D
B A D C
C D B A
D C A B

1

A B C D
B C D A
C D A B
D A B C

2

A B C D
B D A C
C A D B
D C B A

3

Second Transformation Set:
1 Self-Conjugate Standard Square

A B C D
B A D C
C D A B
D C B A

4

### The 5×5 Latin Squares

First Transformation Set: 25 Standard Squares and their Conjugates

A B C D E
B A E C D
C D A E B
D E B A C
E C D B A

1, 2

A B C D E
B A D E C
C E B A D
D C E B A
E D A C B

3, 4

A B C D E
B A E C D
C E D A B
D C B E A
E D A B C

5, 6

A B C D E
B A D E C
C D E A B
D E B C A
E C A B D

7, 8

A B C D E
B C D E A
C E B A D
D A E B C
E D A C B

9, 10

A B C D E
B C D E A
C E A B D
D A E C B
E D B A C

11, 12

A B C D E
B C E A D
C D B E A
D E A C B
E A D B C

13, 14

A B C D E
B C E A D
C A D E B
D E B C A
E D A B C

15, 16

A B C D E
B C D E A
C A E B D
D E A C B
E D B A C

17, 18

A B C D E
B C A E D
C E D A B
D A E B C
E D B C A

19, 20

A B C D E
B C A E D
C D E B A
D E B A C
E A D C B

21, 22

A B C D E
B D E C A
C A B E D
D E A B C
E C D A B

23, 24

A B C D E
B D A E C
C E D B A
D C E A B
E A B C D

25, 26

A B C D E
B D E C A
C A D E B
D E A B C
E C B A D

27, 28

A B C D E
B D A E C
C E D B A
D A E C B
E C B A D

29, 30

A B C D E
B D E A C
C E D B A
D C A E B
E A B C D

31, 32

A B C D E
B D A E C
C E B A D
D A E C B
E C D B A

33, 34

A B C D E
B D E A C
C E A B D
D C B E A
E A D C B

35, 36

A B C D E
B E D A C
C A B E D
D C E B A
E D A C B

37, 38

A B C D E
B E D A C
C A E B D
D C A E B
E D B C A

39, 40

A B C D E
B E A C D
C D E B A
D A B E C
E C D A B

41, 42

A B C D E
B E D C A
C A E B D
D C A E B
E D B A C

43, 44

A B C D E
B E D C A
C D E A B
D A B E C
E C A B D

45, 46

A B C D E
B E A C D
C D B E A
D C E A B
E A D B C

47, 48

A B C D E
B E D A C
C D A E B
D C E B A
E A B C D

49, 50

Second Transformation Set: 6 Self-Conjugate Standard Squares

A B C D E
B C E A D
C E D B A
D A B E C
E D A C B

51

A B C D E
B C D E A
C D E A B
D E A B C
E A B C D

52

A B C D E
B D E C A
C E B A D
D C A E B
E A D B C

53

A B C D E
B D A E C
C A E B D
D E B C A
E C D A B

54

A B C D E
B E D A C
C D B E A
D A E C B
E C A B D

55

A B C D E
B E A C D
C A D E B
D C E B A
E D B A C

56

When constructing squares for use in experimental arrangements use the randomization procedure described in the Introduction.

## TABLE XV. LATIN SQUARES—*continued*

### The 6×6 Latin Squares

Five conjugate pairs of transformation sets, and twelve sets containing conjugates. The numbers above the squares indicate the number in each set, *s* standing for self-conjugate and *c* for conjugate. Each pair of conjugate sets can be turned into the following transformation set by interchanging rows or columns with letters.

I

1080, 1080

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | F | A | D | E |
| C | F | B | E | A | D |
| D | E | A | B | F | C |
| E | A | D | F | C | B |
| F | D | E | C | B | A |

0001-1080
1081-2160

II

180*s*+450+450*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | F | E | A | D |
| C | F | B | A | D | E |
| D | E | A | B | F | C |
| E | A | D | F | C | B |
| F | D | E | C | B | A |

2161-3240

III

540+540*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | A | F | E | C | D |
| C | F | B | A | D | E |
| D | C | E | B | F | A |
| E | D | A | F | B | C |
| F | E | D | C | A | B |

3241-4320

IV

540+540*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | A | E | F | C | D |
| C | F | B | A | D | E |
| D | E | A | B | F | C |
| E | D | F | C | B | A |
| F | C | D | E | A | B |

4321-5400

V

540, 540

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | A | E | C | F | D |
| C | F | B | A | D | E |
| D | E | F | B | C | A |
| E | D | A | F | B | C |
| F | C | D | E | A | B |

5401-5940
5941-6480

VI

90*s*+225+225*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | A | F | E | C | D |
| C | F | B | A | D | E |
| D | E | A | B | F | C |
| E | C | D | F | B | A |
| F | D | E | C | A | B |

6481-7020

VII

270+270*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | D | E | F | A |
| C | E | A | F | B | D |
| D | F | B | A | C | E |
| E | D | F | B | A | C |
| F | A | E | C | D | B |

7021-7560

VIII

360, 360

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | A | E | F | C | D |
| C | F | A | E | D | B |
| D | C | B | A | F | E |
| E | D | F | C | B | A |
| F | E | D | B | A | C |

7561-7920
7921-8280

IX

180+180*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | A | E | F | C | D |
| C | F | A | B | D | E |
| D | E | B | A | F | C |
| E | D | F | C | B | A |
| F | C | D | E | A | B |

8281-8640

X

60*s*+60+60*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | F | A | D | E |
| C | F | B | E | A | D |
| D | A | E | B | F | C |
| E | D | A | F | C | B |
| F | E | D | C | B | A |

8641-8820

XI

120, 120

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | A | F | D | E |
| C | A | B | E | F | D |
| D | F | E | B | A | C |
| E | D | F | C | B | A |
| F | E | D | A | C | B |

8821-8940
8941-9060

XII

60*s*+30+30*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | A | E | F | D |
| C | A | B | F | D | E |
| D | E | F | B | A | C |
| E | F | D | A | C | B |
| F | D | E | C | B | A |

9061-9180

XIII

60*s*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | A | F | D | E |
| C | A | B | E | F | D |
| D | F | E | B | A | C |
| E | D | F | A | C | B |
| F | E | D | C | B | A |

9181-9240

XIV

20+20*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | A | E | F | D |
| C | A | B | F | D | E |
| D | F | E | B | A | C |
| E | D | F | C | B | A |
| F | E | D | A | C | B |

9241-9280

XV

36, 36

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | A | F | E | D | C |
| C | D | A | B | F | E |
| D | F | E | A | C | B |
| E | C | B | F | A | D |
| F | E | D | C | B | A |

9281-9316
9317-9352

XVI

6*s*+15+15*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | A | E | C | F | D |
| C | E | A | F | D | B |
| D | C | F | A | B | E |
| E | F | D | B | A | C |
| F | D | B | E | C | A |

9353-9388

XVII

10+10*c*

| | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | F |
| B | C | A | F | D | E |
| C | A | B | E | F | D |
| D | E | F | A | B | C |
| E | F | D | C | A | B |
| F | D | E | B | C | A |

9389-9408

### Examples of 7×7 Latin Squares

| | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G |
| B | E | A | G | F | D | C |
| C | F | G | B | D | A | E |
| D | G | E | F | C | B | A |
| E | D | B | C | A | G | F |
| F | C | D | A | G | E | B |
| G | A | F | E | B | C | D |

(*a*)

| | | | | | | |
|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G |
| B | E | A | G | F | D | C |
| C | F | G | B | D | A | E |
| D | G | E | F | B | C | A |
| E | D | B | C | A | G | F |
| F | C | D | A | G | E | B |
| G | A | F | E | C | B | D |

(*b*)

| | | | | | | |
|---|---|---|---|---|---|---|
| $A_1$ | $B_2$ | $C_3$ | $D_4$ | $E_5$ | $F_6$ | $G_7$ |
| $B_3$ | $F_7$ | $E_6$ | $G_1$ | $C_4$ | $A_2$ | $D_5$ |
| $C_7$ | $D_1$ | $A_5$ | $E_2$ | $B_6$ | $G_4$ | $F_3$ |
| $D_6$ | $C_5$ | $G_2$ | $A_3$ | $F_1$ | $E_7$ | $B_4$ |
| $E_4$ | $G_6$ | $B_1$ | $F_5$ | $A_7$ | $D_3$ | $C_2$ |
| $F_2$ | $A_4$ | $D_7$ | $C_6$ | $G_3$ | $B_5$ | $E_1$ |
| $G_5$ | $E_3$ | $F_4$ | $B_7$ | $D_2$ | $C_1$ | $A_6$ |

(*c*)

| | | | | | | |
|---|---|---|---|---|---|---|
| $A_1$ | $B_2$ | $C_3$ | $D_4$ | $E_5$ | $F_6$ | $G_7$ |
| $B_3$ | $C_7$ | $D_6$ | $E_1$ | $F_4$ | $G_2$ | $A_5$ |
| $C_6$ | $D_5$ | $E_2$ | $F_3$ | $G_1$ | $A_7$ | $B_4$ |
| $D_2$ | $E_4$ | $F_7$ | $G_6$ | $A_3$ | $B_5$ | $C_1$ |
| $E_7$ | $F_1$ | $G_5$ | $A_2$ | $B_6$ | $C_4$ | $D_3$ |
| $F_5$ | $G_3$ | $A_4$ | $B_7$ | $C_2$ | $D_1$ | $E_6$ |
| $G_4$ | $A_6$ | $B_1$ | $C_5$ | $D_7$ | $E_3$ | $F_2$ |

(*d*)

When constructing squares for use in experimental arrangements use the randomization procedure described in the Introduction.

## TABLE XV. LATIN SQUARES—*continued*

Examples of Squares from 8×8 to 12×12

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H |
| B | C | A | E | F | D | H | G |
| C | A | D | G | H | E | F | B |
| D | F | G | C | A | H | B | E |
| E | H | B | F | G | C | A | D |
| F | D | H | A | B | G | E | C |
| G | E | F | H | C | B | D | A |
| H | G | E | B | D | A | C | F |

8×8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H | I |
| B | C | E | G | D | I | F | A | H |
| C | D | F | A | H | G | I | E | B |
| D | H | A | B | F | E | C | I | G |
| E | G | B | I | C | H | D | F | A |
| F | I | H | E | B | D | A | G | C |
| G | F | I | C | A | B | H | D | E |
| H | E | G | F | I | A | B | C | D |
| I | A | D | H | G | C | E | B | F |

9×9

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H | I | J |
| B | G | A | E | H | C | F | I | J | D |
| C | H | J | G | F | B | E | A | D | I |
| D | A | G | I | J | E | C | B | F | H |
| E | F | H | J | I | G | A | D | B | C |
| F | E | B | C | D | I | J | G | H | A |
| G | I | F | B | A | D | H | J | C | E |
| H | C | I | F | G | J | D | E | A | B |
| I | J | D | A | C | H | B | F | E | G |
| J | D | E | H | B | A | I | C | G | F |

10×10

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H | I | J | K |
| B | A | J | I | D | C | F | K | H | G | E |
| C | K | H | A | B | I | J | F | D | E | G |
| D | C | G | J | I | K | E | B | F | A | H |
| E | J | B | G | K | H | D | C | A | I | F |
| F | E | I | C | G | A | K | J | B | H | D |
| G | F | D | B | H | J | A | I | E | K | C |
| H | I | K | F | A | D | B | E | G | C | J |
| I | D | E | H | J | B | C | G | K | F | A |
| J | G | A | K | F | E | H | D | C | B | I |
| K | H | F | E | C | G | I | A | J | D | B |

11×11

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H | I | J | K | L |
| B | L | G | C | D | J | K | E | H | A | F | I |
| C | K | A | B | F | L | I | D | G | H | J | E |
| D | F | I | A | L | E | C | G | J | B | H | K |
| E | D | F | G | J | K | A | L | C | I | B | H |
| F | H | K | E | G | C | D | B | A | L | I | J |
| G | I | D | F | K | H | J | A | L | C | E | B |
| H | E | L | J | C | A | B | I | K | D | G | F |
| I | J | B | L | H | G | F | K | D | E | A | C |
| J | C | E | K | A | I | H | F | B | G | L | D |
| K | G | J | H | I | B | L | C | E | F | D | A |
| L | A | H | I | B | D | E | J | F | K | C | G |

12×12

## TABLE XVI. COMPLETE SETS OF ORTHOGONAL LATIN SQUARES

3×3

I

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |

II

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 3 | 1 | 2 |
| 2 | 3 | 1 |

4×4

I

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 1 | 4 | 3 |
| 3 | 4 | 1 | 2 |
| 4 | 3 | 2 | 1 |

*AC, BD*

II

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 1 | 2 |
| 4 | 3 | 2 | 1 |
| 2 | 1 | 4 | 3 |

*ACD, BC*

III

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 4 | 3 | 2 | 1 |
| 2 | 1 | 4 | 3 |
| 3 | 4 | 1 | 2 |

*AD, BCD*

5×5

I

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 1 |
| 3 | 4 | 5 | 1 | 2 |
| 4 | 5 | 1 | 2 | 3 |
| 5 | 1 | 2 | 3 | 4 |

II

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 1 | 2 |
| 5 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 5 | 1 |
| 4 | 5 | 1 | 2 | 3 |

III

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 4 | 5 | 1 | 2 | 3 |
| 2 | 3 | 4 | 5 | 1 |
| 5 | 1 | 2 | 3 | 4 |
| 3 | 4 | 5 | 1 | 2 |

IV

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 5 | 1 | 2 | 3 | 4 |
| 4 | 5 | 1 | 2 | 3 |
| 3 | 4 | 5 | 1 | 2 |
| 2 | 3 | 4 | 5 | 1 |

7×7

I

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 |

II

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 6 | 7 | 1 | 2 | 3 | 4 | 5 |

III

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 5 | 6 | 7 | 1 | 2 | 3 | 4 |

IV

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 5 | 6 | 7 | 1 | 2 | 3 |

V

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 2 | 3 | 4 | 5 | 6 | 7 | 1 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |

VI

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 4 | 5 | 6 | 7 | 1 | 2 | 3 |
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 2 | 3 | 4 | 5 | 6 | 7 | 1 |

## TABLE XVI. COMPLETE SETS OF ORTHOGONAL LATIN SQUARES—*continued*

### 8×8

I

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

*AD*, *BE*, *CF*

II

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |

*AEF*, *BD*, *CE*

III

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 |
| 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |

*ADE*, *BEF*, *CD*

IV

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |

*ADEF*, *BDE*, *CEF*

V

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |

*ADF*, *BDEF*, *CDE*

VI

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |

*AF*, *BDF*, *CDEF*

VII

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 1 | 2 | 7 | 8 | 5 | 6 |
| 6 | 5 | 8 | 7 | 2 | 1 | 4 | 3 |
| 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 4 | 3 | 2 | 1 | 8 | 7 | 6 | 5 |
| 5 | 6 | 7 | 8 | 1 | 2 | 3 | 4 |
| 7 | 8 | 5 | 6 | 3 | 4 | 1 | 2 |

*AE*, *BF*, *CDF*

### 9×9

I

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |

*AC*(*J*), *BD*(*J*),
*AC*(*J*)×*BD*(*J*).

II

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |

*BC*(*J*), *AD*(*I*),
*BC*(*J*)×*AD*(*I*).

III

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |

*ABC*(*Y*), *ABD*(*W*),
*BCD*(*Y*), *ACD*(*Z*).

IV

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |

*ABC*(*X*), *ABD*(*Y*),
*ACD*(*X*), *BCD*(*Z*).

V

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |

*AC*(*I*), *BD*(*I*),
*AC*(*I*)×*BD*(*I*).

VI

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |

*BC*(*I*), *AD*(*J*),
*BC*(*I*)×*AD*(*J*).

VII

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |

*ABC*(*Z*), *ABD*(*X*),
*BCD*(*X*), *ACD*(*W*).

VIII

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |

*ABC*(*W*), *ABD*(*Z*),
*ACD*(*Y*), *BCD*(*W*).

## TABLE XVII. Combinatorial Solutions

| | | |
|---|---|---|
| 1 | $r$ 5 $k$ 3<br>$v$ 6 $b$ 10<br>$\lambda$ 2 | Use 2, deleting any set of five varieties occurring in the same block |
| 3 | $r$ 6 $k$ 4<br>$v$ 10 $b$ 15<br>$\lambda$ 2 | Use 5, deleting any set of six varieties occurring in the same block |
| 5 | $rk$ 6<br>$vb$ 16<br>$\lambda$ 2 | Dicyclic solution<br>$a_1 a_2 a_3 b_1 c_4 d_1$ |
| 7† | $r$ 7<br>$v$ 15<br>$k$ 3<br>$b$ 35<br>$\lambda$ 1 | *abc ade afg ahi ajk alm*<br>*dhl bhj bik bmo bln beg*<br>*ejo cmn clo cdg cef cij*<br>*fkm fio djn ekn dim dko*<br>*gin gkl ehm fjl gho fhn*<br>——— ——— ——— ——— ———<br>*ano bdf chk eil gjm* |
| 11 | $r$ 8 $k$ 4<br>$v$ 9 $b$ 18<br>$\lambda$ 3 | Cyclic solution in two families<br>*abce*<br>*adfi* |
| 15 | $r$ 9<br>$v$ 10<br>$k$ 3<br>$b$ 30<br>$\lambda$ 2 | $a_1 d_1 b_2$ $e_2 a_2 b_2$<br>$b_2 c_1 b_1$ $a_1 e_2 d_1$<br>$e_2 b_2 e_1$ $b_1 c_1 e_2$<br>Cyclic permutation (*abcde*) |
| 18 | $r$ 9 $k$ 3<br>$v$ 19 $b$ 57<br>$\lambda$ 1 | Cyclic solution in three families<br>*agk, bcn, dfi* |
| 20 | $rk$ 9<br>$vb$ 25<br>$\lambda$ 3 | *a b c d e f g h i*  *n x e q l k c g w*<br>*b h j e s p n l u*  *o u l f k i p r c*<br>*c g o m j e v p y*  *p i g o t s w x h*<br>*d m x c h j u w r*  *q a b p u c x y t*<br>*e d y u w q s o f*  *r c s a o w b v n*<br>*f q t j n m i c s*  *s k d h c v y t l*<br>*g r m l d a q s p*  *t j n g a d o u k*<br>*h t r k q b m e o*  *u n h i v y r q g*<br>*i y p w b n k d m*  *v w a t i u l m e*<br>*j l w y g r t f b*  *w p k v f h a j q*<br>*k s u x m g f b v*  *x f v r p t e n d*<br>*l v q b x o d i j*  *y e i s r x j k a*<br>*m o f n y l h a x* |
| 23 | $rk$ 9 $\lambda$ 2<br>$vb$ 37 | Cyclic solution<br>*abdhryzDJ* |
| 26 | $r$ 10 $k$ 7 $\lambda$ 3<br>$v$ 21 $b$ 30 | Use 27, deleting any set of ten varieties occurring in the same block |
| 27 | $rk$ 10<br>$vb$ 31<br>$\lambda$ 3 | $a_1 a_6 b_2 b_5 c_3 c_4 d_3 d_5 d_6 e$<br>$a_2 a_5 b_3 b_4 c_1 c_6 d_3 d_5 d_6 f$<br>$a_3 a_4 b_1 b_6 c_2 c_5 d_3 d_5 d_6 g$<br>$a_1 a_2 a_4 b_1 b_2 b_4 c_1 c_2 c_4 d_7$<br>Cyclic permutation (1234567)<br>$a_1 a_2 a_3 a_4 a_5 a_6 a_7 e f g$ Permutation (*abc*) |
| 2 | $rk$ 5 $\lambda$ 2<br>$vb$ 11 | Cyclic solution<br>*aefgi* |
| 4 | $r$ 6 $k$ 3<br>$v$ 13 $b$ 26<br>$\lambda$ 1 | Cyclic solution in two families<br>*aci*<br>*bfe* |
| 6† | $r$ 7<br>$v$ 8<br>$k$ 4<br>$b$ 14<br>$\lambda$ 3 | *abcd abgh acfh adfg*<br>*efgh cdef bdeg bceh*<br>——— ——— ———<br>*abef aceg adeh*<br>*cdgh bdfh bcfg* |
| 9 | $rk$ 7 $\lambda$ 3<br>$vb$ 15 | Cyclic solution<br>*abcefik* |
| 13 | $r$ 8 $k$ 4<br>$v$ 25 $b$ 50<br>$\lambda$ 1 | Dicyclic solution in two families<br>$a_1 a_2 b_1 e_5$<br>$a_1 a_3 c_1 d_4$ |
| 16 | $r$ 9 $k$ 5<br>$v$ 10 $b$ 18<br>$\lambda$ 4 | Use 19, deleting any set of nine varieties occurring in the same block |
| 17 | $r$ 9 $k$ 6<br>$v$ 16 $b$ 24<br>$\lambda$ 3 | Use 20, deleting any set of nine varieties occurring in the same block |
| 19 | $rk$ 9 $\lambda$ 4<br>$vb$ 19 | Cyclic solution<br>*acefghkno* |
| 21† | $r$ 9<br>$v$ 28<br>$k$ 4<br>$b$ 63<br>$\lambda$ 1 | Invariant variety I, and 27 others $a_1^1 \ldots a_3^3$, $b_1^1 \ldots b_3^3$, $c_1^1 \ldots c_3^3$. Blocks $a_2^1 a_3^1 b_1^2 b_1^3$, $a_2^3 b_2^2 a_3^2 b_3^3$, I $a_1^1 b_1^1 c_1^1$. Cyclic permutation (*abc*) generates 7 blocks in a replication; (123) in suffixes and indices generates the 9 replications. |
| 22 | $r$ 9 $k$ 7<br>$v$ 28 $b$ 36<br>$\lambda$ 2 | Use 23, deleting any set of nine varieties occurring in the same block |
| 25† | $r$ 10<br>$v$ 21<br>$k$ 3<br>$b$ 70<br>$\lambda$ 1 | *abc def ghi jkl mno pqr stu*<br>*ado bek cip fqt gls hmr jnu*<br>*aeq bdn cgk fjs hpt ior lmu*<br>*afi bgp chu dqs ejm kot lnr*<br>*agu bmq cjr dhk eps flo int*<br>*ahj brs coq dlp eiu fkn gmt*<br>*akr bjt cel dim fpu gnq hos*<br>*alt bfh cns dru ego ijq kmp*<br>*ams bil cdt ehn fgr jop kqu*<br>*anp bou cfm dgj ert hlq iks* |
| 29 | $r$ 10 $k$ 5<br>$v$ 41 $b$ 82<br>$\lambda$ 1 | Cyclic solution in two families<br>*ajprK*<br>*asEFI* |

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† Divisible into groups, each containing a complete replication, No. 25 by rows, Nos. 6, 7 and 21 as indicated. (See Introduction.)

$$rv = bk, \ \lambda = r(k-1)/(v-1).$$

## TABLE XVIII. Index by Number of Replications

| Replications (*r*). | Varieties (*v*). | Blocks (*b*). | Units per block (*k*). | Reference. |
|---|---|---|---|---|
| 3 | 4 | 6 | 2 | *u* |
| | 4 | 4 | 3 | *u* |
| | 7 | 7 | 3 | *o.s.* |
| 4 | 5 | 10 | 2 | *u* |
| | 5 | 5 | 4 | *u* |
| | 7 | 7 | 4 | *o.s.* |
| | 9 | 12 | 3 | *o.s.* |
| | 13 | 13 | 4 | *o.s.* |
| 5 | 6 | 15 | 2 | *u* |
| | 6 | 10 | 3 | 1 |
| | 6 | 6 | 5 | *u* |
| | 11 | 11 | 5 | 2 |
| | 16 | 20 | 4 | *o.s.* |
| | 21 | 21 | 5 | *o.s.* |
| 6 | 5 | 10 | 3 | *u* |
| | 7 | 21 | 2 | *u* |
| | 7 | 7 | 6 | *u* |
| | 10 | 15 | 4 | 3 |
| | 11 | 11 | 6 | 2 |
| | 13 | 26 | 3 | 4 |
| | 16 | 16 | 6 | 5 |
| | 25 | 30 | 5 | *o.s.* |
| | 31 | 31 | 6 | *o.s.* |
| 7 | 8 | 28 | 2 | *u* |
| | 8 | 14 | 4 | 6 |
| | 8 | 8 | 7 | *u* |
| | 15 | 35 | 3 | 7 |
| | 15 | 21 | 5 | 8* |
| | 15 | 15 | 7 | 9 |
| | 22 | 22 | 7 | 10* |
| 8 | 9 | 36 | 2 | *u* |
| | 9 | 18 | 4 | 11 |
| | 9 | 12 | 6 | *o.s.* |
| | 9 | 9 | 8 | *u* |
| | 15 | 15 | 8 | 9 |
| | 21 | 28 | 6 | 12* |
| | 25 | 50 | 4 | 13 |
| | 29 | 29 | 8 | 14* |
| | 49 | 56 | 7 | *o.s.* |
| | 57 | 57 | 8 | *o.s.* |
| 9 | 10 | 45 | 2 | *u* |
| | 10 | 30 | 3 | 15 |
| | 10 | 18 | 5 | 16 |
| | 10 | 15 | 6 | 3 |
| | 10 | 10 | 9 | *u* |
| | 13 | 13 | 9 | *o.s.* |
| | 16 | 24 | 6 | 17 |
| 9 | 19 | 57 | 3 | 18 |
| | 19 | 19 | 9 | 19 |
| | 25 | 25 | 9 | 20 |
| | 28 | 63 | 4 | 21 |
| | 28 | 36 | 7 | 22 |
| | 37 | 37 | 9 | 23 |
| | 46 | 69 | 6 | 24† |
| | 64 | 72 | 8 | *o.s.* |
| | 73 | 73 | 9 | *o.s.* |
| 10 | 6 | 15 | 4 | *u* |
| | 9 | 18 | 5 | 11 |
| | 11 | 55 | 2 | *u* |
| | 11 | 11 | 10 | *u* |
| | 16 | 16 | 10 | 5 |
| | 19 | 19 | 10 | 19 |
| | 21 | 70 | 3 | 25 |
| | 21 | 30 | 7 | 26 |
| | 31 | 31 | 10 | 27 |
| | 36 | 45 | 8 | 28* |
| | 41 | 82 | 5 | 29 |
| | 46 | 46 | 10 | 30* |
| | 51 | 85 | 6 | 31† |
| | 81 | 90 | 9 | *o.s.* |
| | 91 | 91 | 10 | *o.s.* |

## TABLE XIX. Index by Number of Units in a Block

| Units per block (*k*). | Blocks (*b*). | Varieties (*v*). | Replications (*r*). | Reference. |
|---|---|---|---|---|
| 3 | 4 | 4 | 3 | *u* |
| | 7 | 7 | 3 | *o.s.* |
| | 10 | 5 | 6 | *u* |
| | 10 | 6 | 5 | 1 |
| | 12 | 9 | 4 | *o.s.* |
| | 26 | 13 | 6 | 4 |
| | 30 | 10 | 9 | 15 |
| | 35 | 15 | 7 | 7 |
| | 57 | 19 | 9 | 18 |
| | 70 | 21 | 10 | 25 |
| 4 | 5 | 5 | 4 | *u* |
| | 7 | 7 | 4 | *o.s.* |
| | 13 | 13 | 4 | *o.s.* |
| | 14 | 8 | 7 | 6 |
| | 15 | 6 | 10 | *u* |
| | 15 | 10 | 6 | 3 |
| | 18 | 9 | 8 | 11 |
| | 20 | 16 | 5 | *o.s.* |
| | 50 | 25 | 8 | 13 |
| | 63 | 28 | 9 | 21 |
| 5 | 6 | 6 | 5 | *u* |
| | 11 | 11 | 5 | 2 |
| 5 | 18 | 9 | 10 | 11 |
| | 18 | 10 | 9 | 16 |
| | 21 | 15 | 7 | 8* |
| | 21 | 21 | 5 | *o.s.* |
| | 30 | 25 | 6 | *o.s.* |
| | 82 | 41 | 10 | 29 |
| 6 | 7 | 7 | 6 | *u* |
| | 11 | 11 | 6 | 2 |
| | 12 | 9 | 8 | *o.s.* |
| | 15 | 10 | 9 | 3 |
| | 16 | 16 | 6 | 5 |
| | 24 | 16 | 9 | 17 |
| | 28 | 21 | 8 | 12* |
| | 31 | 31 | 6 | *o.s.* |
| | 69 | 46 | 9 | 24† |
| | 85 | 51 | 10 | 31† |
| 7 | 8 | 8 | 7 | *u* |
| | 15 | 15 | 7 | 9 |
| | 22 | 22 | 7 | 10* |
| | 30 | 21 | 10 | 26 |
| | 36 | 28 | 9 | 22 |
| | 56 | 49 | 8 | *o.s.* |
| 8 | 9 | 9 | 8 | *u* |
| | 15 | 15 | 8 | 9 |
| | 29 | 29 | 8 | 14* |
| | 45 | 36 | 10 | 28* |
| | 57 | 57 | 8 | *o.s.* |
| | 72 | 64 | 9 | *o.s.* |
| 9 | 10 | 10 | 9 | *u* |
| | 13 | 13 | 9 | *o.s.* |
| | 19 | 19 | 9 | 19 |
| | 25 | 25 | 9 | 20 |
| | 37 | 37 | 9 | 23 |
| | 73 | 73 | 9 | *o.s.* |
| | 90 | 81 | 10 | *o.s.* |
| 10 | 11 | 11 | 10 | *u* |
| | 16 | 16 | 10 | 5 |
| | 19 | 19 | 10 | 19 |
| | 31 | 31 | 10 | 27 |
| | 46 | 46 | 10 | 30* |
| | 91 | 91 | 10 | *o.s.* |

* Proved non-existent.

† No solution yet discovered.

## TABLE XX. SCORES FOR ORDINAL (OR RANKED) DATA

The mean deviations of the 1st, 2nd, 3rd . . . largest members of samples of different sizes; zero and negative values omitted.

| Ordinal number. | Size of Sample<br>— | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | ·56 | ·85 | 1·03 | 1·16 | 1·27 | 1·35 | 1·42 | 1·49 | 1·54 |
| 2 | | | | ·30 | ·50 | ·64 | ·76 | ·85 | ·93 | 1·00 |
| 3 | | | | | | ·20 | ·35 | ·47 | ·57 | ·66 |
| 4 | | | | | | | | ·15 | ·27 | ·38 |
| 5 | | | | | | | | | | ·12 |
| | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 1 | 1·59 | 1·63 | 1·67 | 1·70 | 1·74 | 1·76 | 1·79 | 1·82 | 1·84 | 1·87 |
| 2 | 1·06 | 1·12 | 1·16 | 1·21 | 1·25 | 1·28 | 1·32 | 1·35 | 1·38 | 1·41 |
| 3 | ·73 | ·79 | ·85 | ·90 | ·95 | ·99 | 1·03 | 1·07 | 1·10 | 1·13 |
| 4 | ·46 | ·54 | ·60 | ·66 | ·71 | ·76 | ·81 | ·85 | ·89 | ·92 |
| 5 | ·22 | ·31 | ·39 | ·46 | ·52 | ·57 | ·62 | ·67 | ·71 | ·75 |
| 6 | | ·10 | ·19 | ·27 | ·34 | ·39 | ·45 | ·50 | ·55 | ·59 |
| 7 | | | | ·09 | ·17 | ·23 | ·30 | ·35 | ·40 | ·45 |
| 8 | | | | | | ·08 | ·15 | ·21 | ·26 | ·31 |
| 9 | | | | | | | | ·07 | ·13 | ·19 |
| 10 | | | | | | | | | | ·06 |
| | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 1 | 1·89 | 1·91 | 1·93 | 1·95 | 1·97 | 1·98 | 2·00 | 2·01 | 2·03 | 2·04 |
| 2 | 1·43 | 1·46 | 1·48 | 1·50 | 1·52 | 1·54 | 1·56 | 1·58 | 1·60 | 1·62 |
| 3 | 1·16 | 1·19 | 1·21 | 1·24 | 1·26 | 1·29 | 1·31 | 1·33 | 1·35 | 1·36 |
| 4 | ·95 | ·98 | 1·01 | 1·04 | 1·07 | 1·09 | 1·11 | 1·14 | 1·16 | 1·18 |
| 5 | ·78 | ·82 | ·85 | ·88 | ·91 | ·93 | ·96 | ·98 | 1·00 | 1·03 |
| 6 | ·63 | ·67 | ·70 | ·73 | ·76 | ·79 | ·82 | ·85 | ·87 | ·89 |
| 7 | ·49 | ·53 | ·57 | ·60 | ·64 | ·67 | ·70 | ·73 | ·75 | ·78 |
| 8 | ·36 | ·41 | ·45 | ·48 | ·52 | ·55 | ·58 | ·61 | ·64 | ·67 |
| 9 | ·24 | ·29 | ·33 | ·37 | ·41 | ·44 | ·48 | ·51 | ·54 | ·57 |
| 10 | ·12 | ·17 | ·22 | ·26 | ·30 | ·34 | ·38 | ·41 | ·44 | ·47 |
| 11 | | ·06 | ·11 | ·16 | ·20 | ·24 | ·28 | ·32 | ·35 | ·38 |
| 12 | | | | ·05 | 10 | ·14 | ·19 | ·22 | ·26 | ·29 |
| 13 | | | | | | ·05 | ·09 | ·13 | ·17 | ·21 |
| 14 | | | | | | | | ·04 | ·09 | ·12 |
| 15 | | | | | | | | | | ·04 |

Tests of psychological preference and some other experimental data suffice to place a series of magnitudes in order of preference, without supplying metrical values. Analyses of variance, correlations, etc., can be carried out on such data by using the normal scores, appropriate to each position in order, in a sample of the size observed. Ties may be scored with the means of the ordinal values involved, but in such cases the sums of squares given in Table XXI will require correction.

## TABLE XXI. SUMS OF SQUARES OF MEAN DEVIATIONS TABULATED

| *n* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | *n* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 0·6272 | 1·4450 | 2·3018 | 3·1912 | 4·1250 | 5·0452 | 5·9646 | 6·9656 | 7·9320 | 10 |
| 11 | 8·8892 | 9·8662 | 10·8104 | 11·7846 | 12·8232 | 13·6600 | 14·7258 | 15·7454 | 16·6864 | 17·7144 | 20 |
| 21 | 18·6242 | 19·6862 | 20·6176 | 21·6040 | 22·6352 | 23·5470 | 24·5992 | 25·5808 | 26·5806 | 27·5454 | 30 |
| 31 | 28·5730 | 29·5960 | 30·5562 | 31·5152 | 32·5618 | 33·5166 | 34·5346 | 35·4840 | 36·4414 | 37·4288 | 40 |
| 41 | 38·4660 | 39·4654 | 40·4366 | 41·5788 | 42·5194 | 43·4922 | 44·4548 | 45·3626 | 46·5110 | 47·3830 | 50 |

## TABLE XX. SCORES FOR ORDINAL (OR RANKED) DATA—*continued*

| Ordinal number. | Size of Sample | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 1 | 2·06 | 2·07 | 2·08 | 2·09 | 2·11 | 2·12 | 2·13 | 2·14 | 2·15 | 2·16 |
| 2 | 1·63 | 1·65 | 1·66 | 1·68 | 1·69 | 1·70 | 1·72 | 1·73 | 1·74 | 1·75 |
| 3 | 1·38 | 1·40 | 1·42 | 1·43 | 1·45 | 1·46 | 1·48 | 1·49 | 1·50 | 1·52 |
| 4 | 1·20 | 1·22 | 1·23 | 1·25 | 1·27 | 1·28 | 1·30 | 1·32 | 1·33 | 1·34 |
| 5 | 1·05 | 1·07 | 1·09 | 1·11 | 1·12 | 1·14 | 1·16 | 1·17 | 1·19 | 1·20 |
| 6 | ·92 | ·94 | ·96 | ·98 | 1·00 | 1·02 | 1·03 | 1·05 | 1·07 | 1·08 |
| 7 | ·80 | ·82 | ·85 | ·87 | ·89 | ·91 | ·92 | ·94 | ·96 | ·98 |
| 8 | ·69 | ·72 | ·74 | ·76 | ·79 | ·81 | ·83 | ·85 | ·86 | ·88 |
| 9 | ·60 | ·62 | ·65 | ·67 | ·69 | ·72 | ·73 | ·75 | ·77 | ·79 |
| 10 | ·50 | ·53 | ·56 | ·58 | ·60 | ·63 | ·65 | ·67 | ·69 | ·71 |
| 11 | ·41 | ·44 | ·47 | ·50 | ·52 | ·54 | ·57 | ·59 | ·61 | ·63 |
| 12 | ·33 | ·36 | ·39 | ·41 | ·44 | ·47 | ·49 | ·51 | ·54 | ·56 |
| 13 | ·24 | ·28 | ·31 | ·34 | ·36 | ·39 | ·42 | ·44 | ·46 | ·49 |
| 14 | ·16 | ·20 | ·23 | ·26 | ·29 | ·32 | ·34 | ·37 | ·39 | ·42 |
| 15 | ·08 | ·12 | ·15 | ·18 | ·22 | ·24 | ·27 | ·30 | ·33 | ·35 |
| 16 | | ·04 | ·08 | ·11 | ·14 | ·17 | ·20 | ·23 | ·26 | ·28 |
| 17 | | | | ·04 | ·07 | ·10 | ·14 | ·16 | ·19 | ·22 |
| 18 | | | | | | ·03 | ·07 | ·10 | ·13 | ·16 |
| 19 | | | | | | | | ·03 | ·06 | ·09 |
| 20 | | | | | | | | | | ·03 |
| | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 1 | 2·17 | 2·18 | 2·19 | 2·20 | 2·21 | 2·22 | 2·22 | 2·23 | 2·24 | 2·25 |
| 2 | 1·76 | 1·78 | 1·79 | 1·80 | 1·81 | 1·82 | 1·83 | 1·84 | 1·85 | 1·85 |
| 3 | 1·53 | 1·54 | 1·55 | 1·57 | 1·58 | 1·59 | 1·60 | 1·61 | 1·62 | 1·63 |
| 4 | 1·36 | 1·37 | 1·38 | 1·40 | 1·41 | 1·42 | 1·43 | 1·44 | 1·45 | 1·46 |
| 5 | 1·22 | 1·23 | 1·25 | 1·26 | 1·27 | 1·28 | 1·30 | 1·31 | 1·32 | 1·33 |
| 6 | 1·10 | 1·11 | 1·13 | 1·14 | 1·16 | 1·17 | 1·18 | 1·19 | 1·21 | 1·22 |
| 7 | ·99 | 1·01 | 1·02 | 1·04 | 1·05 | 1·07 | 1·08 | 1·09 | 1·11 | 1·12 |
| 8 | ·90 | ·91 | ·93 | ·95 | ·96 | ·98 | ·99 | 1·00 | 1·02 | 1·03 |
| 9 | ·81 | ·83 | ·84 | ·86 | ·88 | ·89 | ·91 | ·92 | ·94 | ·95 |
| 10 | ·73 | ·75 | ·76 | ·78 | ·80 | ·81 | ·83 | ·84 | ·86 | ·87 |
| 11 | ·65 | ·67 | ·69 | ·71 | ·72 | ·74 | ·76 | ·77 | ·79 | ·80 |
| 12 | ·58 | ·60 | ·62 | ·64 | ·65 | ·67 | ·69 | ·70 | ·72 | ·74 |
| 13 | ·51 | ·53 | ·55 | ·57 | ·59 | ·60 | ·62 | ·64 | ·66 | ·67 |
| 14 | ·44 | ·46 | ·48 | ·50 | ·52 | ·54 | ·56 | ·58 | ·59 | ·61 |
| 15 | ·37 | ·40 | ·42 | ·44 | ·46 | ·48 | ·50 | ·52 | ·53 | ·55 |
| 16 | ·31 | ·33 | ·36 | ·38 | ·40 | ·42 | ·44 | ·46 | ·48 | ·49 |
| 17 | ·25 | ·27 | ·29 | ·32 | ·34 | ·36 | ·38 | ·40 | ·42 | ·44 |
| 18 | ·18 | ·21 | ·23 | ·26 | ·28 | ·30 | ·32 | ·34 | ·36 | ·38 |
| 19 | ·12 | ·15 | ·17 | ·20 | ·22 | ·25 | ·27 | ·29 | ·31 | ·33 |
| 20 | ·06 | ·09 | ·12 | ·14 | ·17 | ·19 | ·21 | ·24 | ·26 | ·28 |
| 21 | | ·03 | ·06 | ·09 | ·11 | ·14 | ·16 | ·18 | ·21 | ·23 |
| 22 | | | | ·03 | ·06 | ·08 | ·11 | ·13 | ·15 | ·18 |
| 23 | | | | | | ·03 | ·05 | ·08 | ·10 | ·13 |
| 24 | | | | | | | | ·03 | ·05 | ·08 |
| 25 | | | | | | | | | | ·03 |

# TABLE XXII. INITIAL DIFFERENCES OF POWERS OF NATURAL NUMBERS

These values, the so-called "differences of zero," enter into many combinatorial formulæ, as in Whitworth's *Choice and Chance*. They afford exact solutions of some important sampling problems (Stevens, (12), (13), 1937). In particular if $s$ objects are distributed among $n$ classes, the probabilities of any one object falling into each class being equal, the distribution of $r$, the number of occupied classes, is

$$\frac{1}{n^s}\,\frac{n!}{(n-r)!\,r!}\,\Delta^r 0^s$$

The values tabulated, $\Delta^r 0^s/r!$, are the $r$th differences of powers of natural numbers, advancing from $0^s$, divided by $r!$, for values of $s$ from 2 to 25.

| r \ s | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 3 | 7 | 15 | 31 | 63 | 127 |
| 3 | | 1 | 6 | 25 | 90 | 301 | 966 |
| 4 | | | 1 | 10 | 65 | 350 | 1701 |
| 5 | | | | 1 | 15 | 140 | 1050 |
| 6 | | | | | 1 | 21 | 266 |
| 7 | | | | | | 1 | 28 |
| 8 | | | | | | | 1 |

| r \ s | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|
| 2 | 255 | 511 | 1023 | 2047 | 4095 |
| 3 | 3025 | 9330 | 28501 | 86526 | 2 61625 |
| 4 | 7770 | 34105 | 1 45750 | 6 11501 | 25 32530 |
| 5 | 6951 | 42525 | 2 46730 | 13 79400 | 75 08501 |
| 6 | 2646 | 22827 | 1 79487 | 13 23652 | 93 21312 |
| 7 | 462 | 5880 | 63987 | 6 27396 | 57 15424 |
| 8 | 36 | 750 | 11880 | 1 59027 | 18 99612 |
| 9 | 1 | 45 | 1155 | 22275 | 3 59502 |
| 10 | | 1 | 55 | 1705 | 39325 |
| 11 | | | 1 | 66 | 2431 |
| 12 | | | | 1 | 78 |
| 13 | | | | | 1 |

| r \ s | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|
| 2 | 8191 | 16383 | 32767 | 65535 | 1 31071 |
| 3 | 7 88970 | 23 75101 | 71 41686 | 214 57825 | 644 39010 |
| 4 | 103 91745 | 423 55950 | 1717 98901 | 6943 37290 | 27988 06985 |
| 5 | 400 75035 | 2107 66920 | 10961 90550 | 56527 51651 | 2 89580 95545 |
| 6 | 634 36373 | 4206 93273 | 27349 26558 | 1 75057 49898 | 11 06872 51039 |
| 7 | 493 29280 | 4087 41333 | 32818 82604 | 2 57081 04786 | 19 74624 83400 |
| 8 | 209 12320 | 2166 27840 | 21417 64053 | 2 04159 95028 | 18 90360 65010 |
| 9 | 51 35130 | 671 28490 | 8207 84250 | 95288 22303 | 10 61753 95755 |
| 10 | 7 52752 | 126 62650 | 1937 54990 | 27583 34150 | 3 71121 63803 |
| 11 | 66066 | 14 79478 | 289 36908 | 5120 60978 | 83910 04908 |
| 12 | 3367 | 1 06470 | 27 57118 | 620 22324 | 12563 28866 |
| 13 | 91 | 4550 | 1 65620 | 49 10178 | 1258 54638 |
| 14 | 1 | 105 | 6020 | 2 49900 | 84 08778 |
| 15 | | 1 | 120 | 7820 | 3 67200 |
| 16 | | | 1 | 136 | 9996 |
| 17 | | | | 1 | 153 |
| 18 | | | | | 1 |

## TABLE XXII. INITIAL DIFFERENCES OF POWERS OF NATURAL NUMBERS—*continued*

| r \ s | 19 | 20 | 21 | 22 |
|---|---|---|---|---|
| 2 | 2 62143 | 5 24287 | 10 48575 | 20 97151 |
| 3 | 1934 48101 | 5806 06446 | 17423 43625 | 52280 79450 |
| 4 | 1 12596 66950 | 4 52321 15901 | 18 15090 70050 | 72 77786 23825 |
| 5 | 14 75892 84710 | 74 92060 90500 | 379 12625 68401 | 1913 78219 12055 |
| 6 | 69 30816 01779 | 430 60788 95384 | 2658 56794 62804 | 16330 53393 45225 |
| 7 | 149 29246 34839 | 1114 35540 45652 | 8231 09572 14948 | 60276 23799 67440 |
| 8 | 170 97510 03480 | 1517 09326 62679 | 13251 10153 47084 | 1 14239 90799 91620 |
| 9 | 114 46146 26805 | 1201 12826 44725 | 12327 24764 65204 | 1 24196 33035 33920 |
| 10 | 47 72970 33785 | 591 75849 64655 | 7118 71322 91275 | 83514 37993 77954 |
| 11 | 12 94132 17791 | 190 08424 29486 | 2682 68516 89001 | 36628 25008 70286 |
| 12 | 2 34669 51300 | 41 10166 33391 | 683 30420 30178 | 10882 33560 51137 |
| 13 | 28924 39160 | 6 10686 60380 | 120 49092 18331 | 2249 68618 68481 |
| 14 | 2435 77530 | 63025 24580 | 14 93040 04500 | 329 51652 81331 |
| 15 | 139 16778 | 4523 29200 | 1 30874 62580 | 34 56159 43200 |
| 16 | 5 27136 | 223 50954 | 8099 44464 | 2 60465 74004 |
| 17 | 12597 | 7 41285 | 349 52799 | 14041 42047 |
| 18 | 171 | 15675 | 10 23435 | 533 74629 |
| 19 | 1 | 190 | 19285 | 13 89850 |
| 20 |  | 1 | 210 | 23485 |
| 21 |  |  | 1 | 231 |
| 22 |  |  |  | 1 |

| r \ s | 23 | 24 | 25 |
|---|---|---|---|
| 2 | 41 94303 | 83 88607 | 167 77215 |
| 3 | 1 56863 35501 | 4 70632 00806 | 14 11979 91025 |
| 4 | 291 63425 74750 | 1168 10566 34501 | 4677 12897 38810 |
| 5 | 9641 68881 84100 | 48500 07834 95250 | 2 43668 49741 10751 |
| 6 | 99896 98579 83405 | 6 09023 60360 84530 | 37 02641 70000 02430 |
| 7 | 4 38264 19991 17305 | 31 67746 38518 04540 | 227 83248 29987 16310 |
| 8 | 9 74195 50199 00400 | 82 31828 21583 20505 | 690 22372 11183 68580 |
| 9 | 12 32006 88117 96900 | 120 62257 43260 72500 | 1167 92145 10929 73005 |
| 10 | 9 59340 12973 13460 | 108 25408 17849 31500 | 1203 16339 21753 87500 |
| 11 | 4 86425 13089 51100 | 63 10016 56957 75560 | 802 35590 44384 62660 |
| 12 | 1 67216 27734 83930 | 24 93020 45907 58260 | 362 26262 07848 74680 |
| 13 | 40128 25603 41390 | 6 88883 60579 22000 | 114 48507 33437 44260 |
| 14 | 6862 91758 07115 | 1 36209 10216 41000 | 25 95811 03608 96000 |
| 15 | 847 94044 29331 | 19582 02422 47080 | 4 29939 46553 47200 |
| 16 | 76 23611 27264 | 2067 71824 65555 | 52665 51616 95960 |
| 17 | 4 99169 88803 | 161 09499 36915 | 4806 33313 93110 |
| 18 | 23648 85369 | 9 24849 25445 | 327 56785 94925 |
| 19 | 797 81779 | 38807 39170 | 16 62189 69675 |
| 20 | 18 59550 | 1169 72779 | 62201 94750 |
| 21 | 28336 | 24 54606 | 1685 19505 |
| 22 | 253 | 33902 | 32 00450 |
| 23 | 1 | 276 | 40250 |
| 24 |  | 1 | 300 |
| 25 |  |  | 1 |

# TABLE XXIII. ORTHOGONAL POLYNOMIALS

(Values from $n' = 32$ to $n' = 75$ are due to V. Satakopan)

| 3 | |
|---|---|
| $\xi'_1$ | $\xi'_2$ |
| -1 | +1 |
| 0 | -2 |
| +1 | +1 |
| 2 | 6 |
| 1 | 3 |

| 4 | | |
|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ |
| -3 | +1 | -1 |
| -1 | -1 | +3 |
| +1 | -1 | -3 |
| +3 | +1 | +1 |
| 20 | 4 | 20 |
| 2 | 1 | $\frac{10}{3}$ |

| 5 | | | |
|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ |
| -2 | +2 | -1 | +1 |
| -1 | -1 | +2 | -4 |
| 0 | -2 | 0 | +6 |
| +1 | -1 | -2 | -4 |
| +2 | +2 | +1 | +1 |
| 10 | 14 | 10 | 70 |
| 1 | 1 | $\frac{5}{6}$ | $\frac{35}{12}$ |

| 6 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| -5 | +5 | -5 | +1 | -1 |
| -3 | -1 | +7 | -3 | +5 |
| -1 | -4 | +4 | +2 | -10 |
| +1 | -4 | -4 | +2 | +10 |
| +3 | -1 | -7 | -3 | -5 |
| +5 | +5 | +5 | +1 | +1 |
| 70 | 84 | 180 | 28 | 252 |
| 2 | $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{7}{12}$ | $\frac{21}{10}$ |

| 7 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| -3 | +5 | -1 | +3 | -1 |
| -2 | 0 | +1 | -7 | +4 |
| -1 | -3 | +1 | +1 | -5 |
| 0 | -4 | 0 | +6 | 0 |
| +1 | -3 | -1 | +1 | +5 |
| +2 | 0 | -1 | -7 | -4 |
| +3 | +5 | +1 | +3 | +1 |
| 28 | 84 | 6 | 154 | 84 |
| 1 | 1 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{7}{20}$ |

| 8 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| -7 | +7 | -7 | +7 | -7 |
| -5 | +1 | +5 | -13 | +23 |
| -3 | -3 | +7 | -3 | -17 |
| -1 | -5 | +3 | +9 | -15 |
| +1 | -5 | -3 | +9 | +15 |
| +3 | -3 | -7 | -3 | +17 |
| +5 | +1 | -5 | -13 | -23 |
| +7 | +7 | +7 | +7 | +7 |
| 168 | 168 | 264 | 616 | 2184 |
| 2 | 1 | $\frac{2}{3}$ | $\frac{7}{12}$ | $\frac{7}{10}$ |

| 9 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | -20 | 0 | +18 | 0 |
| +1 | -17 | -9 | +9 | +9 |
| +2 | -8 | -13 | -11 | +4 |
| +3 | +7 | -7 | -21 | -11 |
| +4 | +28 | +14 | +14 | +4 |
| 60 | 2,772 | 990 | 2,002 | 468 |
| 1 | 3 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{3}{20}$ |

| 10 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | -4 | -12 | +18 | +6 |
| +3 | -3 | -31 | +3 | +11 |
| +5 | -1 | -35 | -17 | +1 |
| +7 | +2 | -14 | -22 | -14 |
| +9 | +6 | +42 | +18 | +6 |
| 330 | 132 | 8,580 | 2,860 | 780 |
| 2 | $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{5}{12}$ | $\frac{1}{10}$ |

| 11 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | -10 | 0 | +6 | 0 |
| +1 | -9 | -14 | +4 | +4 |
| +2 | -6 | -23 | -1 | +4 |
| +3 | -1 | -22 | -6 | -1 |
| +4 | +6 | -6 | -6 | -6 |
| +5 | +15 | +30 | +6 | +3 |
| 110 | 858 | 4,290 | 286 | 156 |
| 1 | 1 | $\frac{5}{6}$ | $\frac{1}{12}$ | $\frac{1}{40}$ |

| 12 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | -35 | -7 | +28 | +20 |
| +3 | -29 | -19 | +12 | +44 |
| +5 | -17 | -25 | -13 | +29 |
| +7 | +1 | -21 | -33 | -21 |
| +9 | +25 | -3 | -27 | -57 |
| +11 | +55 | +33 | +33 | +33 |
| 572 | 12,012 | 5,148 | 8,008 | 15,912 |
| 2 | 3 | $\frac{2}{3}$ | $\frac{7}{24}$ | $\frac{3}{20}$ |

| 13 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | -14 | 0 | +84 | 0 |
| +1 | -13 | -4 | +64 | +20 |
| +2 | -10 | -7 | +11 | +26 |
| +3 | -5 | -8 | -54 | +11 |
| +4 | +2 | -6 | -96 | -18 |
| +5 | +11 | 0 | -66 | -33 |
| +6 | +22 | +11 | +99 | +22 |
| 182 | 2,002 | 572 | 68,068 | 6,188 |
| 1 | 1 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{7}{120}$ |

| 14 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | -8 | -24 | +108 | +60 |
| +3 | -7 | -67 | +63 | +145 |
| +5 | -5 | -95 | -13 | +139 |
| +7 | -2 | -98 | -92 | +28 |
| +9 | +2 | -66 | -132 | -132 |
| +11 | +7 | +11 | -77 | -187 |
| +13 | +13 | +143 | +143 | +143 |
| 910 | 728 | 97,240 | 136,136 | 235,144 |
| 2 | $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{7}{12}$ | $\frac{7}{30}$ |

| 15 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | -56 | 0 | +756 | 0 |
| +1 | -53 | -27 | +621 | +675 |
| +2 | -44 | -49 | +251 | +1000 |
| +3 | -29 | -61 | -249 | +751 |
| +4 | -8 | -58 | -704 | -44 |
| +5 | +19 | -35 | -869 | -979 |
| +6 | +52 | +13 | -429 | -1144 |
| +7 | +91 | +91 | +1001 | +1001 |
| 280 | 37,128 | 39,780 | 6,466,460 | 10,581,480 |
| 1 | 3 | $\frac{5}{6}$ | $\frac{35}{12}$ | $\frac{21}{20}$ |

| 16 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | -21 | -63 | +189 | +45 |
| +3 | -19 | -179 | +129 | +115 |
| +5 | -15 | -265 | +23 | +131 |
| +7 | -9 | -301 | -101 | +77 |
| +9 | -1 | -267 | -201 | -33 |
| +11 | +9 | -143 | -221 | -143 |
| +13 | +21 | +91 | -91 | -143 |
| +15 | +35 | +455 | +273 | +143 |
| 1,360 | 5,712 | 1,007,760 | 470,288 | 201,552 |
| 2 | 1 | $\frac{10}{3}$ | $\frac{7}{12}$ | $\frac{1}{10}$ |

| 17 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | -24 | 0 | +36 | 0 |
| +1 | -23 | -7 | +31 | +55 |
| +2 | -20 | -13 | +17 | +88 |
| +3 | -15 | -17 | -3 | +83 |
| +4 | -8 | -18 | -24 | +36 |
| +5 | +1 | -15 | -39 | -39 |
| +6 | +12 | -7 | -39 | -104 |
| +7 | +25 | +7 | -13 | -91 |
| +8 | +40 | +28 | +52 | +104 |
| 408 | 7,752 | 3,876 | 16,796 | 100,776 |
| 1 | 1 | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{20}$ |

| 18 | | | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | -40 | -8 | +44 | +220 |
| +3 | -37 | -23 | +33 | +583 |
| +5 | -31 | -35 | +13 | +733 |
| +7 | -22 | -42 | -12 | +588 |
| +9 | -10 | -42 | -36 | +156 |
| +11 | +5 | -33 | -51 | -429 |
| +13 | +23 | -13 | -47 | -871 |
| +15 | +44 | +20 | -12 | -676 |
| +17 | +68 | +68 | +68 | +884 |
| 1,938 | 23,256 | 23,256 | 28,424 | 6,953,544 |
| 2 | $\frac{3}{2}$ | $\frac{1}{3}$ | $\frac{1}{12}$ | $\frac{3}{10}$ |

## TABLE XXIII. ORTHOGONAL POLYNOMIALS—*continued*

|  |  | 19 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −30 | 0 | +396 | 0 |
| +1 | −29 | −44 | +352 | +44 |
| +2 | −26 | −83 | +227 | +74 |
| +3 | −21 | −112 | +42 | +79 |
| +4 | −14 | −126 | −168 | +54 |
| +5 | −5 | −120 | −354 | +3 |
| +6 | +6 | −89 | −453 | −58 |
| +7 | +19 | −28 | −388 | −98 |
| +8 | +34 | +68 | −68 | −68 |
| +9 | +51 | +204 | +612 | +102 |
| 570 |  | 213,180 |  | 89,148 |
|  | 13,566 |  | 2,288,132 |  |
| 1 | 1 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{1}{40}$ |

|  |  | 20 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | −33 | −99 | +1188 | +396 |
| +3 | −31 | −287 | +948 | +1076 |
| +5 | −27 | −445 | +503 | +1441 |
| +7 | −21 | −553 | −77 | +1351 |
| +9 | −13 | −591 | −687 | +771 |
| +11 | −3 | −539 | −1187 | −187 |
| +13 | +9 | −377 | −1402 | −1222 |
| +15 | +23 | −85 | −1122 | −1802 |
| +17 | +39 | +357 | −102 | −1122 |
| +19 | +57 | +969 | +1938 | +1938 |
| 2,660 |  | 4,903,140 |  | 31,201,800 |
|  | 17,556 |  | 22,881,320 |  |
| 2 | 1 | $\frac{10}{3}$ | $\frac{35}{24}$ | $\frac{7}{20}$ |

|  |  | 21 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −110 | 0 | +594 | 0 |
| +1 | −107 | −54 | +540 | +1404 |
| +2 | −98 | −103 | +385 | +2444 |
| +3 | −83 | −142 | +150 | +2819 |
| +4 | −62 | −166 | −130 | +2354 |
| +5 | −35 | −170 | −406 | +1063 |
| +6 | −2 | −149 | −615 | −788 |
| +7 | +37 | −98 | −680 | −2618 |
| +8 | +82 | −12 | −510 | −3468 |
| +9 | +133 | +114 | 0 | −1938 |
| +10 | +190 | +285 | +969 | +3876 |
| 770 |  | 432,630 |  | 121,687,020 |
|  | 201,894 |  | 5,720,330 |  |
| 1 | 3 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{21}{40}$ |

|  |  | 22 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | −20 | −12 | +702 | +390 |
| +3 | −19 | −35 | +585 | +1079 |
| +5 | −17 | −55 | +365 | +1509 |
| +7 | −14 | −70 | +70 | +1554 |
| +9 | −10 | −78 | −258 | +1158 |
| +11 | −5 | −77 | −563 | +363 |
| +13 | +1 | −65 | −775 | −663 |
| +15 | +8 | −40 | −810 | −1598 |
| +17 | +16 | 0 | −570 | −1938 |
| +19 | +25 | +57 | +57 | −969 |
| +21 | +35 | +133 | +1197 | +2261 |
| 3,542 |  | 96,140 |  | 40,562,340 |
|  | 7,084 |  | 8,748,740 |  |
| 2 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{7}{12}$ | $\frac{7}{30}$ |

|  |  | 23 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −44 | 0 | +858 | 0 |
| +1 | −43 | −13 | +793 | +65 |
| +2 | −40 | −25 | +605 | +116 |
| +3 | −35 | −35 | +315 | +141 |
| +4 | −28 | −42 | −42 | +132 |
| +5 | −19 | −45 | −417 | +87 |
| +6 | −8 | −43 | −747 | +12 |
| +7 | +5 | −35 | −955 | −77 |
| +8 | +20 | −20 | −950 | −152 |
| +9 | +37 | +3 | −627 | −171 |
| +10 | +56 | +35 | +133 | −76 |
| +11 | +77 | +77 | +1463 | +209 |
| 1,012 |  | 32,890 |  | 340,860 |
|  | 35,420 |  | 13,123,110 |  |
| 1 | 1 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{1}{60}$ |

|  |  | 24 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | −143 | −143 | +143 | +715 |
| +3 | −137 | −419 | +123 | +2005 |
| +5 | −125 | −665 | +85 | +2893 |
| +7 | −107 | −861 | +33 | +3171 |
| +9 | −83 | −987 | −27 | +2721 |
| +11 | −53 | −1023 | −87 | +1551 |
| +13 | −17 | −949 | −137 | −169 |
| +15 | +25 | −745 | −165 | −2071 |
| +17 | +73 | −391 | −157 | −3553 |
| +19 | +127 | +133 | −97 | −3743 |
| +21 | +187 | +847 | +33 | −1463 |
| +23 | +253 | +1771 | +253 | +4807 |
| 4,600 |  | 17,760,600 |  | 177,928,920 |
|  | 394,680 |  | 394,680 |  |
| 2 | 3 | $\frac{10}{3}$ | $\frac{1}{12}$ | $\frac{3}{10}$ |

|  |  | 25 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −52 | 0 | +858 | 0 |
| +1 | −51 | −77 | +803 | +275 |
| +2 | −48 | −149 | +643 | +500 |
| +3 | −43 | −211 | +393 | +631 |
| +4 | −36 | −258 | +78 | +636 |
| +5 | −27 | −285 | −267 | +501 |
| +6 | −16 | −287 | −597 | +236 |
| +7 | −3 | −259 | −857 | −119 |
| +8 | +12 | −196 | −982 | −488 |
| +9 | +29 | −93 | −897 | −753 |
| +10 | +48 | +55 | −517 | −748 |
| +11 | +69 | +253 | +253 | −253 |
| +12 | +92 | +506 | +1518 | +1012 |
| 1,300 |  | 1,480,050 |  | 7,803,900 |
|  | 53,820 |  | 14,307,150 |  |
| 1 | 1 | $\frac{5}{6}$ | $\frac{5}{12}$ | $\frac{1}{20}$ |

|  |  | 26 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | −28 | −84 | +1386 | +330 |
| +3 | −27 | −247 | +1221 | +935 |
| +5 | −25 | [illegible] | +905 | +1381 |
| +7 | −22 | −518 | +466 | +1582 |
| +9 | −18 | −606 | −54 | +1482 |
| +11 | −13 | −649 | −599 | +1067 |
| +13 | −7 | −637 | −1099 | +377 |
| +15 | 0 | −560 | −1470 | −482 |
| +17 | +8 | −408 | −1614 | −1326 |
| +19 | +17 | −171 | −1419 | −1881 |
| +21 | +27 | +161 | −759 | −1771 |
| +23 | +38 | +598 | +506 | −506 |
| +25 | +50 | +1150 | +2530 | +2530 |
| 5,850 |  | 7,803,900 |  | 48,384,180 |
|  | 16,380 |  | 40,060,020 |  |
| 2 | $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{7}{12}$ | $\frac{1}{10}$ |

|  |  | 27 |  |  |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −182 | 0 | +1638 | 0 |
| +1 | −179 | −18 | +1548 | +3960 |
| +2 | −170 | −35 | +1285 | +7304 |
| +3 | −155 | −50 | +870 | +9479 |
| +4 | −134 | −62 | +338 | +10058 |
| +5 | −107 | −70 | −262 | +8803 |
| +6 | −74 | −73 | −867 | +5728 |
| +7 | −35 | −70 | −1400 | +1162 |
| +8 | +10 | −60 | −1770 | −4188 |
| +9 | +61 | −42 | −1872 | −9174 |
| +10 | +118 | −15 | −1587 | −12144 |
| +11 | +181 | +22 | −782 | −10879 |
| +12 | +250 | +70 | +690 | −2530 |
| +13 | +325 | +130 | +2990 | +16445 |
| 1,638 |  | 101,790 |  | 2,032,135,560 |
|  | 712,530 |  | 56,448,210 |  |
| 1 | 3 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{21}{40}$ |

| | | 28 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | −65 | −39 | +936 | +1560 |
| +3 | −63 | −115 | +840 | +4456 |
| +5 | −59 | −185 | +655 | +6701 |
| +7 | −53 | −245 | +395 | +7931 |
| +9 | −45 | −291 | +81 | +7887 |
| +11 | −35 | −319 | −259 | +6457 |
| +13 | −23 | −325 | −590 | +3718 |
| +15 | −9 | −305 | −870 | −22 |
| +17 | +7 | −255 | −1050 | −4182 |
| +19 | +25 | −171 | −1074 | −7866 |
| +21 | +45 | −49 | −879 | −9821 |
| +23 | +67 | +115 | −395 | −8395 |
| +25 | +91 | +325 | +455 | −1495 |
| +27 | +117 | +585 | +1755 | +13455 |
| 7,308 | 95,004 | 2,103,660 | 19,634,160 | 1,354,757,040 |
| 2 | 1 | $\frac{2}{3}$ | $\frac{7}{24}$ | $\frac{7}{20}$ |

| | | 29 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −70 | 0 | +2184 | 0 |
| +1 | −69 | −104 | +2080 | +1768 |
| +2 | −66 | −203 | +1775 | +3298 |
| +3 | −61 | −292 | +1290 | +4373 |
| +4 | −54 | −366 | +660 | +4818 |
| +5 | −45 | −420 | −66 | +4521 |
| +6 | −34 | −449 | −825 | +3454 |
| +7 | −21 | −448 | −1540 | +1694 |
| +8 | −6 | −412 | −2120 | −556 |
| +9 | +11 | −336 | −2460 | −2946 |
| +10 | +30 | −215 | −2441 | −4958 |
| +11 | +51 | −44 | −1930 | −5885 |
| +12 | +74 | +182 | −780 | −4810 |
| +13 | +99 | +468 | +1170 | −585 |
| +14 | +126 | +819 | +4095 | +8190 |
| 2,030 | 113,274 | 4,207,320 | 107,987,880 | 500,671,080 |
| 1 | 1 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{7}{40}$ |

| | | 30 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | −112 | −112 | +12376 | +1768 |
| +3 | −109 | −331 | +11271 | +5083 |
| +5 | −103 | −535 | +9131 | +7753 |
| +7 | −94 | −714 | +6096 | +9408 |
| +9 | −82 | −858 | +2376 | +9768 |
| +11 | −67 | −957 | −1749 | +8679 |
| +13 | −49 | −1001 | −5929 | +6149 |
| +15 | −28 | −980 | −9744 | +2384 |
| +17 | −4 | −884 | −12704 | −2176 |
| +19 | +23 | −703 | −14249 | −6821 |
| +21 | +53 | −427 | −13749 | −10535 |
| +23 | +86 | −46 | −10504 | −11960 |
| +25 | +122 | +450 | −3744 | −9360 |
| +27 | +161 | +1071 | +7371 | −585 |
| +29 | +203 | +1827 | +23751 | +16965 |
| 8,990 | 302,064 | 21,360,240 | 3,671,587,920 | 2,145,733,200 |
| 2 | $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{35}{12}$ | $\frac{3}{10}$ |

| | | 31 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +0 | −80 | 0 | +408 | 0 |
| +1 | −79 | −119 | +391 | +221 |
| +2 | −76 | −233 | +341 | +416 |
| +3 | −71 | −337 | +261 | +561 |
| +4 | −64 | −426 | +156 | +636 |
| +5 | −55 | −495 | +33 | +627 |
| +6 | −44 | −539 | −99 | +528 |
| +7 | −31 | −553 | −229 | +343 |
| +8 | −16 | −532 | −344 | +88 |
| +9 | +1 | −471 | −429 | −207 |
| +10 | +20 | −365 | −467 | −496 |
| +11 | +41 | −209 | −439 | −715 |
| +12 | +64 | +2 | −324 | −780 |
| +13 | +89 | +273 | −99 | −585 |
| +14 | +116 | +609 | +261 | 0 |
| +15 | +145 | +1015 | +783 | +1131 |
| 2,480 | 158,224 | 6,724,520 | 4,034,712 | 9,536,592 |
| 1 | 1 | $\frac{5}{6}$ | $\frac{1}{12}$ | $\frac{1}{60}$ |

| | | 32 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| +1 | −85 | −51 | +459 | +255 |
| +3 | −83 | −151 | +423 | +737 |
| +5 | −79 | −245 | +353 | +1137 |
| +7 | −73 | −329 | +253 | +1407 |
| +9 | −65 | −399 | +129 | +1509 |
| +11 | −55 | −451 | −11 | +1419 |
| +13 | −43 | −481 | −157 | +1131 |
| +15 | −29 | −485 | −297 | +661 |
| +17 | −13 | −459 | −417 | +51 |
| +19 | +5 | −399 | −501 | −627 |
| +21 | +25 | −301 | −531 | −1267 |
| +23 | +47 | −161 | −487 | −1725 |
| +25 | +71 | +25 | −347 | −1815 |
| +27 | +97 | +261 | −87 | −1305 |
| +29 | +125 | +551 | +319 | +87 |
| +31 | +155 | +899 | +899 | +2697 |
| 10,912 | 185,504 | 5,379,616 | 5,379,616 | 54,285,216 |
| 2 | 1 | $\frac{2}{3}$ | $\frac{1}{12}$ | $\frac{1}{30}$ |

| | | 33 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −272 | 0 | +3672 | 0 |
| +1 | −269 | −27 | +3537 | +2565 |
| +2 | −260 | −53 | +3139 | +4864 |
| +3 | −245 | −77 | +2499 | +6649 |
| +4 | −224 | −98 | +1652 | +7708 |
| +5 | −197 | −115 | +647 | +7883 |
| +6 | −164 | −127 | −453 | +7088 |
| +7 | −125 | −133 | −1571 | +5327 |
| +8 | −80 | −132 | −2616 | +2712 |
| +9 | −29 | −123 | −3483 | −519 |
| +10 | +28 | −105 | −4053 | −3984 |
| +11 | +91 | −77 | −4193 | −7139 |
| +12 | +160 | −38 | −3756 | −9260 |
| +13 | +235 | +13 | −2581 | −9425 |
| +14 | +316 | +77 | −493 | −6496 |
| +15 | +403 | +155 | +2697 | +899 |
| +16 | +496 | +248 | +7192 | +14384 |
| 2,992 | 1,947,792 | 417,384 | 348,330,136 | 1,547,128,656 |
| 1 | 3 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{3}{20}$ |

# TABLE XXIII. ORTHOGONAL POLYNOMIALS—*continued*

| | | 34 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 1 | −48 | −144 | +4104 | +6840 |
| 3 | −47 | −427 | +3819 | +19855 |
| 5 | −45 | −695 | +3263 | +30917 |
| 7 | −42 | −938 | +2464 | +38864 |
| 9 | −38 | −1146 | +1464 | +42744 |
| 11 | −33 | −1309 | +319 | +41899 |
| 13 | −27 | −1417 | −901 | +36049 |
| 15 | −20 | −1460 | −2112 | +25376 |
| 17 | −12 | −1428 | −3216 | +10608 |
| 19 | −3 | −1311 | −4101 | −6897 |
| 21 | +7 | −1099 | −4641 | −25067 |
| 23 | +18 | −782 | −4696 | −41032 |
| 25 | +30 | −350 | −4112 | −51040 |
| 27 | +43 | +207 | −2721 | −50373 |
| 29 | +57 | +899 | −341 | −33263 |
| 31 | +72 | +1736 | +3224 | +7192 |
| 33 | +88 | +2728 | +8184 | +79112 |
| 13,090 | 62,832 | 51,477,360 | 456,432,592 | 46,929,569,232 |
| 2 | $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{7}{12}$ | $\frac{7}{10}$ |

| | | 35 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −102 | 0 | +23256 | 0 |
| 1 | −101 | −152 | +22496 | +3800 |
| 2 | −98 | −299 | +20251 | +7250 |
| 3 | −93 | −436 | +16626 | +10021 |
| 4 | −86 | −558 | +11796 | +11826 |
| 5 | −77 | −660 | +6006 | +12441 |
| 6 | −66 | −737 | −429 | +11726 |
| 7 | −53 | −784 | −7124 | +9646 |
| 8 | −38 | −796 | −13624 | +6292 |
| 9 | −21 | −768 | −19404 | +1902 |
| 10 | −2 | −695 | −23869 | −3118 |
| 11 | +19 | −572 | −26354 | −8173 |
| 12 | +42 | −394 | −26124 | −12458 |
| 13 | +67 | −156 | −22374 | −14937 |
| 14 | +94 | +147 | −14229 | −14322 |
| 15 | +123 | +520 | −744 | −9052 |
| 16 | +154 | +968 | +19096 | +2728 |
| 17 | +187 | +1496 | +46376 | +23188 |
| 3,570 | 290,598 | 15,775,320 | 14,834,059,240 | 4,045,652,520 |
| 1 | 1 | $\frac{5}{6}$ | $\frac{35}{12}$ | $\frac{7}{40}$ |

| | | 36 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 1 | −323 | −323 | +2584 | +12920 |
| 3 | −317 | −959 | +2424 | +37640 |
| 5 | −305 | −1565 | +2111 | +59063 |
| 7 | −287 | −2121 | +1659 | +75201 |
| 9 | −263 | −2607 | +1089 | +84381 |
| 11 | −233 | −3003 | +429 | +85371 |
| 13 | −197 | −3289 | −286 | +77506 |
| 15 | −155 | −3445 | −1014 | +60814 |
| 17 | −107 | −3451 | −1706 | +36142 |
| 19 | −53 | −3287 | −2306 | +5282 |
| 21 | +7 | −2933 | −2751 | −28903 |
| 23 | +73 | −2369 | −2971 | −62353 |
| 25 | +145 | −1575 | −2889 | −89685 |
| 27 | +223 | −531 | −2421 | −104067 |
| 29 | +307 | +783 | −1476 | −97092 |
| 31 | +397 | +2387 | +44 | −58652 |
| 33 | +493 | +4301 | +2244 | +23188 |
| 35 | +595 | +6545 | +5236 | +162316 |
| 15,540 | 3,011,652 | 307,618,740 | 191,407,216 | 199,046,103,984 |
| 2 | 3 | $\frac{10}{3}$ | $\frac{7}{24}$ | $\frac{21}{20}$ |

| | | 37 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −114 | 0 | +5814 | 0 |
| 1 | −113 | −34 | +5644 | +680 |
| 2 | −110 | −67 | +5141 | +1304 |
| 3 | −105 | −98 | +4326 | +1819 |
| 4 | −98 | −126 | +3234 | +2178 |
| 5 | −89 | −150 | +1914 | +2343 |
| 6 | −78 | −169 | +429 | +2288 |
| 7 | −65 | −182 | −1144 | +2002 |
| 8 | −50 | −188 | −2714 | +1492 |
| 9 | −33 | −186 | −4176 | +786 |
| 10 | −14 | −175 | −5411 | −64 |
| 11 | +7 | −154 | −6286 | −979 |
| 12 | +30 | −122 | −6654 | −1850 |
| 13 | +55 | −78 | −6354 | −2535 |
| 14 | +82 | −21 | −5211 | −2856 |
| 15 | +111 | +50 | −3036 | −2596 |
| 16 | +142 | +136 | +374 | −1496 |
| 17 | +175 | +238 | +5236 | +748 |
| 18 | +210 | +357 | +11781 | +4488 |
| 4,218 | 383,838 | 932,178 | 980,961,982 | 152,877,192 |
| 1 | 1 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{1}{40}$ |

| | | 38 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 1 | −60 | −36 | +918 | +1530 |
| 3 | −59 | −107 | +867 | +4471 |
| 5 | −57 | −175 | +767 | +7061 |
| 7 | −54 | −238 | +622 | +9086 |
| 9 | −50 | −294 | +438 | +10362 |
| 11 | −45 | −341 | +223 | +10747 |
| 13 | −39 | −377 | −13 | +10153 |
| 15 | −32 | −400 | −258 | +8558 |
| 17 | −24 | −408 | −498 | +6018 |
| 19 | −15 | −399 | −717 | +2679 |
| 21 | −5 | −371 | −897 | −1211 |
| 23 | +6 | −322 | −1018 | −5290 |
| 25 | +18 | −250 | −1058 | −9070 |
| 27 | +31 | −153 | −993 | −11925 |
| 29 | +45 | −29 | −797 | −13079 |
| 31 | +60 | +124 | −442 | −11594 |
| 33 | +76 | +308 | +102 | −6358 |
| 35 | + 93 | +525 | +867 | +3927 |
| 37 | +111 | +777 | +1887 | +20757 |
| 18,278 | 109,668 | 4,496,388 | 25,479,532 | 3,286,859,628 |
| 2 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{12}$ | $\frac{1}{10}$ |

| | | 39 | | |
|---|---|---|---|---|
| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| 0 | −380 | 0 | +1026 | 0 |
| 1 | −377 | −189 | +999 | +5049 |
| 2 | −368 | −373 | +919 | +9724 |
| 3 | −353 | −547 | +789 | +13669 |
| 4 | −332 | −706 | +614 | +16564 |
| 5 | −305 | −845 | +401 | +18143 |
| 6 | −272 | −959 | +159 | +18212 |
| 7 | −233 | −1043 | −101 | +16667 |
| 8 | −188 | −1092 | −366 | +13512 |
| 9 | −137 | −1101 | −621 | +8877 |
| 10 | −80 | −1065 | −849 | +3036 |
| 11 | −17 | −979 | −1031 | −3575 |
| 12 | +52 | −838 | −1146 | −10340 |
| 13 | +127 | −637 | −1171 | −16445 |
| 14 | +208 | −371 | −1081 | −20860 |
| 15 | +295 | −35 | −849 | −22321 |
| 16 | +388 | +376 | −446 | −19312 |
| 17 | +487 | +867 | +159 | −10047 |
| 18 | +592 | +1443 | +999 | +7548 |
| 19 | +703 | +2109 | +2109 | +35853 |
| 4,940 | 4,496,388 | 33,722,910 | 32,224,114 | 9,860,578,884 |
| 1 | 3 | $\frac{5}{6}$ | $\frac{1}{12}$ | $\frac{3}{20}$ |

40

| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
|---|---|---|---|---|
| 1 | −133 | −399 | +39501 | +627 |
| 3 | −131 | −1187 | +37521 | +1837 |
| 5 | −127 | −1945 | +33631 | +2917 |
| 7 | −121 | −2653 | +27971 | +3787 |
| 9 | −113 | −3291 | +20751 | +4377 |
| 11 | −103 | −3839 | +12251 | +4631 |
| 13 | −91 | −4277 | +2821 | +4511 |
| 15 | −77 | −4585 | −7119 | +4001 |
| 17 | −61 | −4743 | −17079 | +3111 |
| 19 | −43 | −4731 | −26499 | +1881 |
| 21 | −23 | −4529 | −34749 | +385 |
| 23 | −1 | −4117 | −41129 | −1265 |
| 25 | +23 | −3475 | −44869 | −2915 |
| 27 | +49 | −2583 | −45129 | −4365 |
| 29 | +77 | −1421 | −40999 | −5365 |
| 31 | +107 | +31 | −31499 | −5611 |
| 33 | +139 | +1793 | −15579 | −4741 |
| 35 | +173 | +3885 | +7881 | −2331 |
| 37 | +209 | +6327 | +40071 | +2109 |
| 39 | +247 | +9139 | +82251 | +9139 |
| 21,320 | 567,112 | 644,482,280 | 49,625,135,560 | 644,482,280 |
| 2 | 1 | $\frac{10}{3}$ | $\frac{35}{12}$ | $\frac{1}{30}$ |

41

| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
|---|---|---|---|---|
| 0 | −140 | 0 | +8778 | 0 |
| 1 | −139 | −209 | +8569 | +4807 |
| 2 | −136 | −413 | +7949 | +9292 |
| 3 | −131 | −607 | +6939 | +13147 |
| 4 | −124 | −786 | +5574 | +16092 |
| 5 | −115 | −945 | +3903 | +17889 |
| 6 | −104 | −1079 | +1989 | +18356 |
| 7 | −91 | −1183 | −91 | +17381 |
| 8 | −76 | −1252 | −2246 | +14936 |
| 9 | −59 | −1281 | −4371 | +11091 |
| 10 | −40 | −1265 | −6347 | +6028 |
| 11 | −19 | −1199 | −8041 | +55 |
| 12 | +4 | −1078 | −9306 | −6380 |
| 13 | +29 | −897 | −9981 | −12675 |
| 14 | +56 | −651 | −9891 | −18060 |
| 15 | +85 | −335 | −8847 | −21583 |
| 16 | +116 | +56 | −6646 | −22096 |
| 17 | +149 | +527 | −3071 | −18241 |
| 18 | +184 | +1083 | +2109 | −8436 |
| 19 | +221 | +1729 | +9139 | +9139 |
| 20 | +260 | +2470 | +18278 | +36556 |
| 5,740 | 641,732 | 47,900,710 | 2,481,256,778 | 10,376,164,708 |
| 1 | 1 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{7}{60}$ |

42

| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
|---|---|---|---|---|
| 1 | −220 | −44 | +9614 | +48070 |
| 3 | −217 | −131 | +9177 | +141151 |
| 5 | −211 | −215 | +8317 | +225181 |
| 7 | −202 | −294 | +7062 | +294546 |
| 9 | −190 | −366 | +5454 | +344262 |
| 11 | −175 | −429 | +3549 | +370227 |
| 13 | −157 | −481 | +1417 | +369473 |
| 15 | −136 | −520 | −858 | +340418 |
| 17 | −112 | −544 | −3178 | +283118 |
| 19 | −85 | −551 | −5431 | +199519 |
| 21 | −55 | −539 | −7491 | +93709 |
| 23 | −22 | −506 | −9218 | −27830 |
| 25 | +14 | −450 | −10458 | −155970 |
| 27 | +53 | −369 | −11043 | −278685 |
| 29 | +95 | −261 | −10791 | −380799 |
| 31 | +140 | −124 | −9506 | −443734 |
| 33 | +188 | +44 | −6978 | −445258 |
| 35 | +239 | +245 | −2983 | −359233 |
| 37 | +293 | +481 | +2717 | −155363 |
| 39 | +350 | +754 | +10374 | +201058 |
| 41 | +410 | +1066 | +20254 | +749398 |
| 24,682 | 1,629,012 | 9,075,924 | 3,084,805,724 | 4,389,117,671,484 |
| 2 | $\frac{3}{2}$ | $\frac{1}{3}$ | $\frac{7}{12}$ | $\frac{21}{10}$ |

43

| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
|---|---|---|---|---|
| 0 | −154 | 0 | +10626 | 0 |
| 1 | −153 | −46 | +10396 | +8740 |
| 2 | −150 | −91 | +9713 | +16948 |
| 3 | −145 | −134 | +8598 | +24113 |
| 4 | −138 | −174 | +7086 | +29766 |
| 5 | −129 | −210 | +5226 | +33501 |
| 6 | −118 | −241 | +3081 | +34996 |
| 7 | −105 | −266 | +728 | +34034 |
| 8 | −90 | −284 | −1742 | +30524 |
| 9 | −73 | −294 | −4224 | +24522 |
| 10 | −54 | −295 | −6599 | +16252 |
| 11 | −33 | −286 | −8734 | +6127 |
| 12 | −10 | −266 | −10482 | −5230 |
| 13 | +15 | −234 | −11682 | −16965 |
| 14 | +42 | −189 | −12159 | −27972 |
| 15 | +71 | −130 | −11724 | −36872 |
| 16 | +102 | −56 | −10174 | −41992 |
| 17 | +135 | +34 | −7292 | −41344 |
| 18 | +170 | +141 | −2847 | −32604 |
| 19 | +207 | +266 | +3406 | −13091 |
| 20 | +246 | +410 | +11726 | +20254 |
| 21 | +287 | +574 | +22386 | +70889 |
| 6,622 | 814,506 | 2,676,234 | 3,815,417,606 | 39,541,600,644 |
| 1 | 1 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{7}{40}$ |

44

| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
|---|---|---|---|---|
| 1 | −161 | −483 | +5796 | +1380 |
| 3 | −159 | −1439 | +5556 | +4060 |
| 5 | −155 | −2365 | +5083 | +6503 |
| 7 | −149 | −3241 | +4391 | +8561 |
| 9 | −141 | −4047 | +3501 | +10101 |
| 11 | −131 | −4763 | +2441 | +11011 |
| 13 | −119 | −5369 | +1246 | +11206 |
| 15 | −105 | −5845 | −42 | +10634 |
| 17 | −89 | −6171 | −1374 | +9282 |
| 19 | −71 | −6327 | −2694 | +7182 |
| 21 | −51 | −6293 | −3939 | +4417 |
| 23 | −29 | −6049 | −5039 | +1127 |
| 25 | −5 | −5575 | −5917 | −2485 |
| 27 | +21 | −4851 | −6489 | −6147 |
| 29 | +49 | −3857 | −6664 | −9512 |
| 31 | +79 | −2573 | −6344 | −12152 |
| 33 | +111 | −979 | −5424 | −13552 |
| 35 | +145 | +945 | −3792 | −13104 |
| 37 | +181 | +3219 | −1329 | −10101 |
| 39 | +219 | +5863 | +2091 | −3731 |
| 41 | +259 | +8897 | +6601 | +6929 |
| 43 | +301 | +12341 | +12341 | +22919 |
| 28,380 | 913,836 | 1,257,829,980 | 1,173,974,648 | 4,162,273,752 |
| 2 | 1 | $\frac{10}{3}$ | $\frac{7}{24}$ | $\frac{1}{20}$ |

45

| $\xi'_1$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
|---|---|---|---|---|
| 0 | −506 | 0 | +9108 | 0 |
| 1 | −503 | −252 | +8928 | +4500 |
| 2 | −494 | −499 | +8393 | +8750 |
| 3 | −479 | −736 | +7518 | +12509 |
| 4 | −458 | −958 | +6328 | +15554 |
| 5 | −431 | −1160 | +4858 | +17689 |
| 6 | −398 | −1337 | +3153 | +18754 |
| 7 | −359 | −1484 | +1268 | +18634 |
| 8 | −314 | −1596 | −732 | +17268 |
| 9 | −263 | −1668 | −2772 | +14658 |
| 10 | −206 | −1695 | −4767 | +10878 |
| 11 | −143 | −1672 | −6622 | +6083 |
| 12 | −74 | −1594 | −8232 | +518 |
| 13 | +1 | −1456 | −9482 | −5473 |
| 14 | +82 | −1253 | −10247 | −11438 |
| 15 | +169 | −980 | −10392 | −16808 |
| 16 | +262 | −632 | −9772 | −20888 |
| 17 | +361 | −204 | −8232 | −22848 |
| 18 | +466 | +309 | −5607 | −21714 |
| 19 | +577 | +912 | −1722 | −16359 |
| 20 | +694 | +1610 | +3608 | −5494 |
| 21 | +817 | +2408 | +10578 | +12341 |
| 22 | +946 | +3311 | +19393 | +38786 |
| 7,590 | 9,203,634 | 92,036,340 | 2,934,936,620 | 12,006,558,900 |
| 1 | 3 | $\frac{5}{6}$ | $\frac{5}{12}$ | $\frac{3}{40}$ |

## TABLE XXIII. ORTHOGONAL POLYNOMIALS—*continued*

| 46 | | | | 47 | | | | 48 | | | | 49 | | | | 50 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| −88 | −264 | +1980 | +3300 | −184 | 0 | +15180 | 0 | −575 | −115 | +16445 | +82225 | −200 | 0 | +17940 | 0 | −104 | −312 | +96876 | +10764 |
| −87 | −787 | +1905 | +9725 | −183 | −55 | +14905 | +3575 | −569 | −343 | +15873 | +242671 | −199 | −299 | +17641 | +9867 | −103 | −931 | +93771 | +31809 |
| −85 | −1295 | +1757 | +15631 | −180 | −109 | +14087 | +6968 | −557 | −565 | +14743 | +391231 | −196 | −593 | +16751 | +19272 | −101 | −1535 | +87631 | +51419 |
| −82 | −1778 | +1540 | +20692 | −175 | −161 | +12747 | +10003 | −539 | −777 | +13083 | +520401 | −191 | −877 | +15291 | +27767 | −98 | −2114 | +78596 | +68684 |
| −78 | −2226 | +1260 | +24612 | −168 | −210 | +10920 | +12516 | −515 | −975 | +10935 | +623307 | −184 | −1146 | +13296 | +34932 | −94 | −2658 | +66876 | +82764 |
| −73 | −2629 | +925 | +27137 | −159 | −255 | +8655 | +14361 | −485 | −1155 | +8355 | +693957 | −175 | −1395 | +10815 | +40389 | −89 | −3157 | +52751 | +92917 |
| −67 | −2977 | +545 | +28067 | −148 | −295 | +6015 | +15416 | −449 | −1313 | +5413 | +727493 | −164 | −1619 | +7911 | +43816 | −83 | −3601 | +36571 | +98527 |
| −60 | −3260 | +132 | +27268 | −135 | −329 | +3077 | +15589 | −407 | −1445 | +2193 | +720443 | −151 | −1813 | +4661 | +44961 | −76 | −3980 | +18756 | +99132 |
| −52 | −3468 | −300 | +24684 | −120 | −356 | −68 | +14824 | −359 | −1547 | −1207 | +670973 | −136 | −1972 | +1156 | +43656 | −68 | −4284 | −204 | +94452 |
| −43 | −3591 | −735 | +20349 | −103 | −375 | −3315 | +13107 | −305 | −1615 | −4675 | +579139 | −119 | −2091 | −2499 | +39831 | −59 | −4503 | −19749 | +84417 |
| −33 | −3619 | −1155 | +14399 | −84 | −385 | −6545 | +10472 | −245 | −1645 | −8085 | +447139 | −100 | −2165 | −6185 | +33528 | −49 | −4627 | −39249 | +69195 |
| −22 | −3542 | −1540 | +7084 | −63 | −385 | −9625 | +7007 | −179 | −1633 | −11297 | +279565 | −79 | −2189 | −9769 | +24915 | −38 | −4646 | −58004 | +49220 |
| −10 | −3350 | −1868 | −1220 | −40 | −374 | −12408 | +2860 | −107 | −1575 | −14157 | +83655 | −56 | −2158 | −13104 | +14300 | −26 | −4550 | −75244 | +25220 |
| +3 | −3033 | −2115 | −9999 | −15 | −351 | −14733 | −1755 | −29 | −1467 | −16497 | −130455 | −31 | −2067 | −16029 | +2145 | −13 | −4329 | −90129 | −1755 |
| +17 | −2581 | −2255 | −18589 | +12 | −315 | −16425 | −6552 | +55 | −1305 | −18135 | −349479 | −4 | −1911 | −18369 | −10920 | +1 | −3973 | −101749 | −30305 |
| +32 | −1984 | −2260 | −26164 | +41 | −265 | −17295 | −11167 | +145 | −1085 | −18875 | −556729 | +25 | −1685 | −19935 | −24083 | +16 | −3472 | −109124 | −58652 |
| +48 | −1232 | −2100 | −31724 | +72 | −200 | −17140 | −15152 | +241 | −803 | −18507 | −731863 | +56 | −1384 | −20524 | −36336 | +32 | −2816 | −111204 | −84612 |
| +65 | −315 | −1743 | −34083 | +105 | −119 | −15743 | −17969 | +343 | −455 | −16807 | −850633 | +89 | −1003 | −19919 | −46461 | +49 | −1995 | −106869 | −105567 |
| +83 | +777 | −1155 | −31857 | +140 | −21 | −12873 | −18984 | +451 | −37 | −13537 | −884633 | +124 | −537 | −17889 | −53016 | +67 | −999 | −94929 | −118437 |
| +102 | +2054 | −300 | −23452 | +177 | +95 | −8285 | −17461 | +565 | +455 | −8445 | −801047 | +161 | +19 | −14189 | −54321 | +86 | +182 | −74124 | −119652 |
| +122 | +3526 | +860 | −7052 | +216 | +230 | −1720 | −12556 | +685 | +1025 | −1265 | −562397 | +200 | +670 | −8560 | −48444 | +106 | +1558 | −43124 | −105124 |
| +143 | +5203 | +2365 | +19393 | +257 | +385 | +7095 | −3311 | +811 | +1677 | +8283 | −126291 | +241 | +1421 | −729 | −33187 | +127 | +3139 | −529 | −70219 |
| +165 | +7095 | +4257 | +58179 | +300 | +561 | +18447 | +11352 | +943 | +2415 | +20493 | +554829 | +284 | +2277 | +9591 | −6072 | +149 | +4935 | +55131 | −9729 |
| | | | | +345 | +759 | +32637 | +32637 | +1081 | +3243 | +35673 | +1533939 | +329 | +3243 | +22701 | +35673 | +172 | +6956 | +125396 | +82156 |
| | | | | | | | | | | | | +376 | +4324 | +38916 | +95128 | +196 | +9212 | +211876 | +211876 |
| 285,384 | 429,502,920 | 143,167,640 | 27,214,866,840 | 1,271,256 | 4,994,220 | 8,518,474,580 | 8,629,104,120 | 12,712,560 | 92,620,080 | 10,301,411,120 | 19,208,385,771,120 | 1,566,040 | 167,230,700 | 12,408,517,940 | 74,451,107,640 | 433,160 | 770,715,400 | 372,255,538,200 | 372,255,538,200 |
| $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{1}{12}$ | $\frac{1}{10}$ | 1 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{1}{20}$ | 3 | $\frac{2}{3}$ | $\frac{7}{12}$ | $\frac{21}{10}$ | 1 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{7}{60}$ | $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{35}{12}$ | $\frac{7}{30}$ |

From this point onwards $\xi'_1$ is omitted. The value of $S(\xi'^2_1)$ is $n'(n'^2-1)/3$ for even $n'$, and $n'(n'^2-1)/12$ for odd $n'$.

## Table XXIII. Orthogonal Polynomials—*continued*

| 51 | | | | 52 | | | | 53 | | | | 54 | | | | 55 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| -650 | 0 | +21060 | 0 | -225 | -135 | +1620 | +2700 | -234 | 0 | +3510 | 0 | -364 | -364 | +26390 | +18850 | -252 | 0 | +142506 | 0 |
| -647 | -324 | +20736 | +7452 | -223 | -403 | +1572 | +7988 | -233 | -70 | +3460 | +2900 | -361 | -1087 | +25665 | +55825 | -251 | -377 | +140621 | +47125 |
| -638 | -643 | +19771 | +14582 | -219 | -665 | +1477 | +12943 | -230 | -139 | +3311 | +5684 | -355 | -1795 | +24229 | +90643 | -248 | -749 | +135001 | +92500 |
| -623 | -952 | +18186 | +21077 | -213 | -917 | +1337 | +17353 | -225 | -206 | +3066 | +8239 | -346 | -2478 | +22110 | +121926 | -243 | -1111 | +125751 | +134417 |
| -602 | -1246 | +16016 | +26642 | -205 | -1155 | +1155 | +21021 | -218 | -270 | +2730 | +10458 | -334 | -3126 | +19350 | +148386 | -236 | -1458 | +113046 | +171252 |
| -575 | -1520 | +13310 | +31009 | -195 | -1375 | +935 | +23771 | -209 | -330 | +2310 | +12243 | -319 | -3729 | +16005 | +168861 | -227 | -1785 | +97131 | +201507 |
| -542 | -1769 | +10131 | +33946 | -183 | -1573 | +682 | +25454 | -198 | -385 | +1815 | +13508 | -301 | -4277 | +12145 | +182351 | -216 | -2087 | +78321 | +223852 |
| -503 | -1988 | +6556 | +35266 | -169 | -1745 | +402 | +25954 | -185 | -434 | +1256 | +14182 | -280 | -4760 | +7854 | +188054 | -203 | -2359 | +57001 | +237167 |
| -458 | -2172 | +2676 | +34836 | -153 | -1887 | +102 | +25194 | -170 | -476 | +646 | +14212 | -256 | -5168 | +3230 | +185402 | -188 | -2596 | +33626 | +240584 |
| -407 | -2316 | -1404 | +32586 | -135 | -1995 | -210 | +23142 | -153 | -510 | 0 | +13566 | -229 | -5491 | -1615 | +174097 | -171 | -2793 | +8721 | +233529 |
| -350 | -2415 | -5565 | +28518 | -115 | -2065 | -525 | +19817 | -134 | -535 | -665 | +12236 | -199 | -5719 | -6555 | +154147 | -152 | -2945 | -17119 | +215764 |
| -287 | -2464 | -9674 | +22715 | -93 | -2093 | -833 | +15295 | -113 | -550 | -1330 | +10241 | -166 | -5842 | -11450 | +125902 | -131 | -3047 | -43229 | +187429 |
| -218 | -2458 | -13584 | +15350 | -69 | -2075 | -1123 | +9715 | -90 | -554 | -1974 | +7630 | -130 | -5850 | -16146 | +90090 | -108 | -3094 | -68874 | +149084 |
| -143 | -2392 | -17134 | +6695 | -43 | -2007 | -1383 | +3285 | -65 | -546 | -2574 | +4485 | -91 | -5733 | -20475 | +47853 | -83 | -3081 | -93249 | +101751 |
| -62 | -2261 | -20149 | -2870 | -15 | -1885 | -1600 | -3712 | -38 | -525 | -3105 | +924 | -49 | -5481 | -24255 | +783 | -56 | -3003 | -115479 | +46956 |
| +25 | -2060 | -22440 | -12848 | +15 | -1705 | -1760 | -10912 | -9 | -490 | -3540 | -2896 | -4 | -5084 | -27290 | -49042 | -27 | -2855 | -134619 | -13229 |
| +118 | -1784 | -23804 | -22616 | +47 | -1463 | -1848 | -17864 | +22 | -440 | -3850 | -6776 | +44 | -4532 | -29370 | -99022 | +4 | -2632 | -149654 | -76144 |
| +217 | -1428 | -24024 | -31416 | +81 | -1155 | -1848 | -24024 | +55 | -374 | -4004 | -10472 | +95 | -3815 | -30271 | -145999 | +37 | -2329 | -159499 | -138499 |
| +322 | -987 | -22869 | -38346 | +117 | -777 | -1743 | -28749 | +90 | -291 | -3969 | -13692 | +149 | -2923 | -29755 | -186221 | +72 | -1941 | -162999 | -196332 |
| +433 | -456 | -20094 | -42351 | +155 | -325 | -1515 | -31291 | +127 | -190 | -3710 | -16093 | +206 | -1846 | -27570 | -215306 | +109 | -1463 | -158929 | -244967 |
| +550 | +170 | -15440 | -42214 | +195 | +205 | -1145 | -30791 | +166 | -70 | -3190 | -17278 | +266 | -574 | -23450 | -228206 | +148 | -890 | -145994 | -278972 |
| +673 | +896 | -8634 | -36547 | +237 | +817 | -613 | -26273 | +207 | +70 | -2370 | -16793 | +329 | +903 | -17115 | -219171 | +189 | -217 | -122829 | -292117 |
| +802 | +1727 | +611 | -23782 | +281 | +1515 | +102 | -16638 | +250 | +231 | -1209 | -14124 | +395 | +2595 | -8271 | -181713 | +232 | +561 | -87999 | -277332 |
| +937 | +2668 | +12596 | -2162 | +327 | +2303 | +1022 | -658 | +295 | +414 | +336 | -8694 | +464 | +4512 | +3390 | -108570 | +277 | +1449 | -39999 | -226665 |
| +1078 | +3724 | +27636 | +30268 | +375 | +3185 | +2170 | +23030 | +342 | +620 | +2310 | +140 | +536 | +6664 | +18190 | +8330 | +324 | +2452 | +22746 | -131240 |
| +1225 | +4900 | +46060 | +75670 | +425 | +4165 | +3570 | +55930 | +391 | +850 | +4760 | +13090 | +611 | +9061 | +36465 | +177905 | +373 | +3575 | +101881 | +18785 |
| | | | | | | | | +442 | +1105 | +7735 | +30940 | +689 | +11713 | +58565 | +409955 | +424 | +4823 | +199121 | +234260 |
| | | | | | | | | | | | | | | | | +477 | +6201 | +316251 | +527085 |
| | 221,375,700 | | 47,861,426,340 | | 162,342,180 | | 26,358,466,680 | | 11,595,870 | | 8,131,867,380 | | 1,321,929,180 | | 1,439,340,526,260 | | 375,842,610 | | 2,398,900,877,100 |
| 17,218,110 | | 17,803,525,740 | | 2,108,340 | | 108,228,120 | | 2,319,174 | | 514,083,570 | | 5,729,724 | | 29,816,847,060 | | 2,791,404 | | 879,596,988,270 | |
| 3 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{3}{40}$ | 1 | $\frac{2}{3}$ | $\frac{1}{24}$ | $\frac{1}{20}$ | 1 | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{40}$ | $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{7}{12}$ | $\frac{3}{10}$ | 1 | $\frac{5}{6}$ | $\frac{35}{12}$ | $\frac{7}{20}$ |

## TABLE XXIII. ORTHOGONAL POLYNOMIALS—*continued*

| 56 | | | | 57 | | | | 58 | | | | 59 | | | | 60 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| -261 | -783 | +30537 | +50895 | -812 | 0 | +32886 | 0 | -140 | -84 | +35154 | +2790 | -290 | 0 | +5394 | 0 | -899 | -899 | +14384 | +14384 |
| -259 | -2339 | +29757 | +150865 | -809 | -81 | +32481 | +163215 | -139 | -251 | +34317 | +8277 | -289 | -434 | +5332 | +1488 | -893 | -2687 | +14064 | +42704 |
| -255 | -3865 | +28211 | +245417 | -800 | -161 | +31273 | +320788 | -137 | -415 | +32657 | +13487 | -286 | -863 | +5147 | +2928 | -881 | -4445 | +13429 | +69689 |
| -249 | -5341 | +25927 | +331079 | -785 | -239 | +29283 | +467203 | -134 | -574 | +30202 | +18242 | -281 | -1282 | +4842 | +4273 | -863 | -6153 | +12489 | +94479 |
| -241 | -6747 | +22947 | +404589 | -764 | -314 | +26546 | +597196 | -130 | -726 | +26994 | +22374 | -274 | -1686 | +4422 | +5478 | -839 | -7791 | +11259 | +116259 |
| -231 | -8063 | +19327 | +462979 | -737 | -385 | +23111 | +705881 | -125 | -869 | +23089 | +25729 | -265 | -2070 | +3894 | +6501 | -809 | -9339 | +9759 | +134277 |
| -219 | -9269 | +15137 | +503659 | -704 | -451 | +19041 | +788876 | -119 | -1001 | +18557 | +28171 | -254 | -2429 | +3267 | +7304 | -773 | -10777 | +8014 | +147862 |
| -205 | -10345 | +10461 | +524501 | -665 | -511 | +14413 | +842429 | -112 | -1120 | +13482 | +29586 | -241 | -2758 | +2552 | +7854 | -731 | -12085 | +6054 | +156442 |
| -189 | -11271 | +5397 | +523923 | -620 | -564 | +9318 | +863544 | -104 | -1224 | +7962 | +29886 | -226 | -3052 | +1762 | +8124 | -683 | -13243 | +3914 | +159562 |
| -171 | -12027 | +57 | +500973 | -569 | -609 | +3861 | +850107 | -95 | -1311 | +2109 | +29013 | -209 | -3306 | +912 | +8094 | -629 | -14231 | +1634 | +156902 |
| -151 | -12593 | -5433 | +455413 | -512 | -645 | -1839 | +801012 | -85 | -1379 | -3951 | +26943 | -190 | -3515 | +19 | +7752 | -569 | -15029 | -741 | +148295 |
| -129 | -12949 | -10933 | +387803 | -449 | -671 | -7649 | +716287 | -74 | -1426 | -10078 | +23690 | -169 | -3674 | -898 | +7095 | -503 | -15617 | -3161 | +133745 |
| -105 | -13075 | -16289 | +299585 | -380 | -686 | -13422 | +597220 | -62 | -1450 | -16118 | +19310 | -146 | -3778 | -1818 | +6130 | -431 | -15975 | -5571 | +113445 |
| -79 | -12951 | -21333 | +193167 | -305 | -689 | -18997 | +446485 | -49 | -1449 | -21903 | +13905 | -121 | -3822 | -2718 | +4875 | -353 | -16083 | -7911 | +87795 |
| -51 | -12557 | -25883 | +72007 | -224 | -679 | -24199 | +268268 | -35 | -1421 | -27251 | +7627 | -94 | -3801 | -3573 | +3360 | -269 | -15921 | -10116 | +57420 |
| -21 | -11873 | -29743 | -59303 | -137 | -655 | -28839 | +68393 | -20 | -1364 | -31966 | +682 | -65 | -3710 | -4356 | +1628 | -179 | -15469 | -12116 | +23188 |
| +11 | -10879 | -32703 | -194953 | -44 | -616 | -32714 | -145552 | -4 | -1276 | -35838 | -6666 | -34 | -3544 | -5038 | -264 | -83 | -14707 | -13836 | -13772 |
| +45 | -9555 | -34539 | -327831 | +55 | -561 | -35607 | -364089 | +13 | -1155 | -38643 | -14091 | -1 | -3298 | -5588 | -2244 | +19 | -13615 | -15196 | -52052 |
| +81 | -7881 | -35013 | -449439 | +160 | -489 | -37287 | -575724 | +31 | -999 | -40143 | -21201 | +34 | -2967 | -5973 | -4224 | +127 | -12173 | -16111 | -89947 |
| +119 | -5837 | -33873 | -549809 | +271 | -399 | -37509 | -766821 | +50 | -806 | -40086 | -27534 | +71 | -2546 | -6158 | -6099 | +241 | -10361 | -16491 | -125437 |
| +159 | -3403 | -30853 | -617419 | +388 | -290 | -36014 | -921476 | +70 | -574 | -38206 | -32554 | +110 | -2030 | -6106 | -7746 | +361 | -8159 | -16241 | -156169 |
| +201 | -559 | -25673 | -639109 | +511 | -161 | -32529 | -1021391 | +91 | -301 | -34223 | -35647 | +151 | -1414 | -5778 | -9023 | +487 | -5547 | -15261 | -179439 |
| +245 | +2715 | -18039 | -599997 | +640 | -11 | -26767 | -1045748 | +113 | +15 | -27843 | -36117 | +194 | -693 | -5133 | -9768 | +619 | -2505 | -13446 | -192174 |
| +291 | +6439 | -7643 | -483395 | +775 | +161 | -18427 | -971083 | +136 | +376 | -18758 | -33182 | +239 | +138 | -4128 | -9798 | +757 | +987 | -10686 | -190914 |
| +339 | +10633 | +5837 | -270725 | +916 | +356 | -7194 | -771160 | +160 | +784 | -6646 | -25970 | +286 | +1084 | -2718 | -8908 | +901 | +4949 | -6866 | -171794 |
| +389 | +15317 | +22737 | +58565 | +1063 | +575 | +7261 | -416845 | +185 | +1241 | +8829 | -13515 | +335 | +2150 | -856 | -6870 | +1051 | +9401 | -1866 | -130526 |
| +441 | +20511 | +43407 | +527085 | +1216 | +819 | +25281 | +124020 | +211 | +1749 | +28017 | +5247 | +386 | +3341 | +1507 | -3432 | +1207 | +14363 | +4439 | -62381 |
| +495 | +26235 | +68211 | +1159587 | +1375 | +1089 | +47223 | +886743 | +238 | +2310 | +51282 | +31482 | +439 | +4662 | +4422 | +1683 | +1369 | +19855 | +12179 | +37829 |
| | | | | +1540 | +1386 | +73458 | +1909908 | +266 | +2926 | +79002 | +66462 | +494 | +6118 | +7942 | +8778 | +1537 | +25897 | +21489 | +175769 |
| | | | | | | | | | | | | +551 | +7714 | +12122 | +18183 | +1711 | +32509 | +32509 | +357599 |
| 3,054,744 | 6,822,988,920 | 41,392,799,448 | 11,706,636,280,248 | 30,038,316 | 19,310,346 | 48,556,937,814 | 32,021,093,354,796 | 910,252 | 87,254,156 | 56,802,455,556 | 39,097,794,084 | 3,966,098 | 614,745,190 | 1,352,439,418 | 2,950,776,912 | 38,826,012 | 11,065,413,420 | 9,835,923,040 | 1,150,802,995,680 |
| 1 | $\frac{10}{3}$ | $\frac{7}{12}$ | $\frac{7}{10}$ | 3 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{21}{20}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{7}{12}$ | $\frac{1}{30}$ | 1 | $\frac{5}{6}$ | $\frac{1}{12}$ | $\frac{1}{120}$ | 3 | $\frac{10}{3}$ | $\frac{5}{24}$ | $\frac{3}{20}$ |

TABLE XXIII. ORTHOGONAL POLYNOMIALS—*continued*

| 61 | | | | 62 | | | | 63 | | | | 64 | | | | 65 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| −310 | 0 | +43152 | 0 | −160 | −96 | +45936 | +76560 | −992 | 0 | +49104 | 0 | −341 | −1023 | +52173 | +86955 | −352 | 0 | +278256 | 0 |
| −309 | −464 | +42688 | +5104 | −159 | −287 | +44979 | +227447 | −989 | −99 | +48609 | +244035 | −339 | −3059 | +51153 | +258485 | −351 | −527 | +275621 | +13175 |
| −306 | −923 | +41303 | +10054 | −157 | −475 | +43079 | +371677 | −980 | −197 | +47131 | +481168 | −335 | −5065 | +49127 | +422917 | −348 | −1049 | +267751 | +26000 |
| −301 | −1372 | +39018 | +14699 | −154 | −658 | +40264 | +504952 | −965 | −293 | +44691 | +704623 | −329 | −7021 | +46123 | +575659 | −343 | −1561 | +254751 | +38131 |
| −294 | −1806 | +35868 | +18894 | −150 | −834 | +36576 | +623184 | −944 | −386 | +41324 | +907876 | −321 | −8907 | +42183 | +712329 | −336 | −2058 | +236796 | +49236 |
| −285 | −2220 | +31902 | +22503 | −145 | −1001 | +32071 | +722579 | −917 | −475 | +37079 | +1084781 | −311 | −10703 | +37363 | +828839 | −327 | −2535 | +214131 | +59001 |
| −274 | −2609 | +27183 | +25402 | −139 | −1157 | +26819 | +799721 | −884 | −559 | +32019 | +1229696 | −299 | −12389 | +31733 | +921479 | −316 | −2987 | +187071 | +67136 |
| −261 | −2968 | +21788 | +27482 | −132 | −1300 | +20904 | +851656 | −845 | −637 | +26221 | +1337609 | −285 | −13945 | +25377 | +987001 | −303 | −3409 | +156001 | +73381 |
| −246 | −3292 | +15808 | +28652 | −124 | −1428 | +14424 | +875976 | −800 | −708 | +19776 | +1404264 | −269 | −15351 | +18393 | +1022703 | −288 | −3796 | +121376 | +77512 |
| −229 | −3576 | +9348 | +28842 | −115 | −1539 | +7491 | +870903 | −749 | −771 | +12789 | +1426287 | −251 | −16587 | +10893 | +1026513 | −271 | −4143 | +83721 | +79347 |
| −210 | −3815 | +2527 | +28006 | −105 | −1631 | +231 | +835373 | −692 | −825 | +5379 | +1401312 | −231 | −17633 | +3003 | +997073 | −252 | −4445 | +43631 | +78752 |
| −189 | −4004 | −4522 | +26125 | −94 | −1702 | −7216 | +769120 | −629 | −869 | −2321 | +1328107 | −209 | −18469 | −5137 | +933823 | −231 | −4697 | +1771 | +75647 |
| −166 | −4138 | −11652 | +23210 | −82 | −1750 | −14696 | +672760 | −560 | −902 | −10164 | +1206700 | −185 | −19075 | −13373 | +837085 | −208 | −4894 | −41124 | +70012 |
| −141 | −4212 | −18702 | +19305 | −69 | −1773 | −22041 | +547875 | −485 | −923 | −17989 | +1038505 | −159 | −19431 | −21537 | +708147 | −183 | −5031 | −84249 | +61893 |
| −114 | −4221 | −25497 | +14490 | −55 | −1769 | −29069 | +397097 | −404 | −931 | −25621 | +826448 | −131 | −19517 | −29447 | +549347 | −156 | −5103 | −126729 | +51408 |
| −85 | −4160 | −31848 | +8884 | −40 | −1736 | −35584 | +224192 | −317 | −925 | −32871 | +575093 | −101 | −19313 | −36907 | +364157 | −127 | −5105 | −167619 | +38753 |
| −54 | −4024 | −37552 | +2648 | −24 | −1672 | −41376 | +34144 | −224 | −904 | −39536 | +290768 | −69 | −18799 | −43707 | +157267 | −96 | −5032 | −205904 | +24208 |
| −21 | −3808 | −42392 | −4012 | −7 | −1575 | −46221 | −166761 | −125 | −867 | −45399 | −18309 | −35 | −17955 | −49623 | −65331 | −63 | −4879 | −240499 | +8143 |
| +14 | −3507 | −46137 | −10842 | +11 | −1443 | −49881 | −370851 | −20 | −813 | −50229 | −341904 | +1 | −16761 | −54417 | −296259 | −28 | −4641 | −270249 | −8976 |
| +51 | −3116 | −48542 | −17537 | +30 | −1274 | −52104 | −568984 | +91 | −741 | −53781 | −667641 | +39 | −15197 | −57837 | −526669 | +9 | −4313 | −293929 | −26581 |
| +90 | −2630 | −49348 | −23738 | +50 | −1066 | −52624 | −750464 | +208 | −650 | −55796 | −980876 | +79 | −13243 | −59617 | −746159 | +48 | −3890 | −310244 | −43996 |
| +131 | −2044 | −48282 | −29029 | +71 | −817 | −51161 | −902957 | +331 | −539 | −56001 | −1264571 | +121 | −10879 | −59477 | −942689 | +89 | −3367 | −317829 | −60431 |
| +174 | −1353 | −45057 | −32934 | +93 | −525 | −47421 | −1012407 | +460 | −407 | −54109 | −1499168 | +165 | −8085 | −57123 | −1102497 | +132 | −2739 | −315249 | −74976 |
| +219 | −552 | −39372 | −34914 | +116 | −188 | −41096 | −1062952 | +595 | −253 | −49819 | −1662463 | +211 | −4841 | −52247 | −1210015 | +177 | −2001 | −300999 | −86595 |
| +266 | +364 | −30912 | −34364 | +140 | +196 | −31864 | −1036840 | +736 | −76 | −42816 | −1729480 | +259 | −1127 | −44527 | −1247785 | +224 | −1148 | −273504 | −94120 |
| +315 | +1400 | −19348 | −30610 | +165 | +629 | −19389 | −914345 | +883 | +125 | −32771 | −1672345 | +309 | +3077 | −33627 | −1196375 | +273 | −175 | −231119 | −96245 |
| +366 | +2561 | −4337 | −22906 | +191 | +1113 | −3321 | −673683 | +1036 | +351 | −19341 | −1460160 | +361 | +7791 | −19197 | −1034295 | +324 | +923 | −172129 | −91520 |
| +419 | +3852 | +14478 | −10431 | +218 | +1650 | +16704 | −290928 | +1195 | +603 | −2169 | −1058877 | +415 | +13035 | −873 | −737913 | +377 | +2151 | −94749 | −78345 |
| +474 | +5278 | +37468 | +7714 | +246 | +2242 | +41064 | +260072 | +1360 | +882 | +19116 | −431172 | +471 | +18829 | +21723 | −281371 | +432 | +3514 | +2876 | −54964 |
| +531 | +6844 | +65018 | +32509 | +275 | +2891 | +70151 | +1007779 | +1531 | +1189 | +44899 | +463681 | +529 | +25193 | +48983 | +363499 | +489 | +5017 | +122671 | −19459 |
| +590 | +8555 | +97527 | +65018 | +305 | +3599 | +104371 | +1983049 | +1708 | +1525 | +75579 | +1669936 | +589 | +32147 | +81313 | +1227259 | +548 | +6665 | +266631 | +30256 |
| | | | | | | | | +1891 | +1891 | +111569 | +3235501 | +651 | +39711 | +119133 | +2342949 | +609 | +8453 | +436821 | +96441 |
| | | | | | | | | | | | | | | | | +672 | +10416 | +635376 | +181536 |
| | 776,520,240 | | 38,360,099,856 | | 139,238,112 | | 35,981,773,664,928 | | 38,939,472 | | 96,582,655,626,912 | | 17,392,964,160 | | 51,066,691,480,896 | | 1,211,804,880 | | 309,120,408,480 |
| 4,685,898 | | 89,506,899,664 | | 1,270,752 | | 103,639,568,032 | | 49,559,328 | | 119,721,569,968 | | 5,957,952 | | 137,984,182,336 | | 6,438,432 | | 3,967,045,242,160 | |
| 1 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{1}{40}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{7}{12}$ | $\frac{7}{10}$ | 3 | $\frac{1}{6}$ | $\frac{7}{12}$ | $\frac{21}{20}$ | 1 | $\frac{10}{3}$ | $\frac{7}{12}$ | $\frac{7}{10}$ | 1 | $\frac{5}{6}$ | $\frac{35}{12}$ | $\frac{1}{20}$ |

## TABLE XXIII. ORTHOGONAL POLYNOMIALS—*continued (also overleaf)*

| 66 | | | | 67 | | | | 68 | | | | 69 | | | | 70 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| -544 | -544 | +8432 | +42160 | -374 | 0 | +8976 | 0 | -385 | -231 | +33264 | +2640 | -1190 | 0 | +70686 | 0 | -204 | -612 | +373626 | +124542 |
| -541 | -1627 | +8277 | +125395 | -373 | -112 | +8896 | +2480 | -383 | -691 | +32688 | +7856 | -1187 | -594 | +70092 | +175824 | -203 | -1831 | +367521 | +370777 |
| -535 | -2695 | +7969 | +205393 | -370 | -223 | +8657 | +4898 | -379 | -1145 | +31543 | +12881 | -1178 | -1183 | +68317 | +347504 | -201 | -3035 | +355381 | +608507 |
| -526 | -3738 | +7512 | +280056 | -365 | -332 | +8262 | +7193 | -373 | -1589 | +29843 | +17591 | -1163 | -1762 | +65382 | +510959 | -198 | -4214 | +337346 | +832202 |
| -514 | -4746 | +6912 | +347376 | -358 | -438 | +7716 | +9306 | -365 | -2019 | +27609 | +21867 | -1142 | -2326 | +61322 | +662234 | -194 | -5358 | +313626 | +1036542 |
| -499 | -5709 | +6177 | +405471 | -349 | -540 | +7026 | +11181 | -355 | -2431 | +24869 | +25597 | -1115 | -2870 | +56186 | +797563 | -189 | -6457 | +284501 | +1216501 |
| -481 | -6617 | +5317 | +452621 | -338 | -637 | +6201 | +12766 | -343 | -2821 | +21658 | +28678 | -1082 | -3389 | +50037 | +913432 | -183 | -7501 | +250321 | +1367431 |
| -460 | -7460 | +4344 | +487304 | -325 | -728 | +5252 | +14014 | -329 | -3185 | +18018 | +31018 | -1043 | -3878 | +42952 | +1006642 | -176 | -8480 | +211506 | +1485146 |
| -436 | -8228 | +3272 | +508232 | -310 | -812 | +4192 | +14884 | -313 | -3519 | +13998 | +32538 | -998 | -4332 | +35022 | +1074372 | -168 | -9384 | +168546 | +1566006 |
| -409 | -8911 | +2117 | +514387 | -293 | -888 | +3036 | +15342 | -295 | -3819 | +9654 | +33174 | -947 | -4746 | +26352 | +1114242 | -159 | -10203 | +122001 | +1607001 |
| -379 | -9499 | +897 | +505057 | -274 | -955 | +1801 | +15362 | -275 | -4081 | +5049 | +32879 | -890 | -5115 | +17061 | +1124376 | -149 | -10927 | +72501 | +1605835 |
| -346 | -9982 | -368 | +479872 | -253 | -1012 | +506 | +14927 | -253 | -4301 | +253 | +31625 | -827 | -5434 | +7282 | +1103465 | -138 | -11546 | +20746 | +1561010 |
| -310 | -10350 | -1656 | +438840 | -230 | -1058 | -828 | +14030 | -229 | -4475 | -4657 | +29405 | -758 | -5698 | -2838 | +1050830 | -126 | -12050 | -32494 | +1471910 |
| -271 | -10593 | -2943 | +382383 | -205 | -1092 | -2178 | +12675 | -203 | -4599 | -9597 | +26235 | -683 | -5902 | -13138 | +966485 | -113 | -12429 | -86379 | +1338885 |
| -229 | -10701 | -4203 | +311373 | -178 | -1113 | -3519 | +10878 | -175 | -4669 | -14476 | +22156 | -602 | -6041 | -23443 | +851200 | -99 | -12673 | -139999 | +1163335 |
| -184 | -10664 | -5408 | +227168 | -149 | -1120 | -4824 | +8668 | -145 | -4681 | -19196 | +17236 | -515 | -6110 | -33564 | +706564 | -84 | -12772 | -192374 | +947794 |
| -136 | -10472 | -6528 | +131648 | -118 | -1112 | -6064 | +6088 | -113 | -4631 | -23652 | +11572 | -422 | -6104 | -43298 | +535048 | -68 | -12716 | -242454 | +696014 |
| -85 | -10115 | -7531 | +27251 | -85 | -1088 | -7208 | +3196 | -79 | -4515 | -27732 | +5292 | -323 | -6018 | -52428 | +340068 | -51 | -12495 | -289119 | +413049 |
| -31 | -9583 | -8383 | -82991 | -50 | -1047 | -8223 | +66 | -43 | -4329 | -31317 | -1443 | -218 | -5847 | -60723 | +126048 | -33 | -12099 | -331179 | +105339 |
| +26 | -8866 | -9048 | -195416 | -13 | -988 | -9074 | -3211 | -5 | -4069 | -34281 | -8437 | -107 | -5586 | -67938 | -101517 | -14 | -11518 | -367374 | -219206 |
| +86 | -7954 | -9488 | -305696 | +26 | -910 | -9724 | -6526 | +35 | -3731 | -36491 | -15457 | +10 | -5230 | -73814 | -335998 | +6 | -10742 | -396374 | -551122 |
| +149 | -6837 | -9663 | -408801 | +67 | -812 | -10134 | -9751 | +77 | -3311 | -37807 | -22231 | +133 | -4774 | -78078 | -569569 | +27 | -9761 | -416779 | -879307 |
| +215 | -5505 | -9531 | -498963 | +110 | -693 | -10263 | -12738 | +121 | -2805 | -38082 | -28446 | +262 | -4213 | -80443 | -793144 | +49 | -8565 | -427119 | -1190937 |
| +284 | -3948 | -9048 | -569640 | +155 | -552 | -10068 | -15318 | +167 | -2209 | -37162 | -33746 | +397 | -3542 | -80608 | -996314 | +72 | -7144 | -425854 | -1471382 |
| +356 | -2156 | -8168 | -613480 | +202 | -388 | -9504 | -17300 | +215 | -1519 | -34886 | -37730 | +538 | -2756 | -78258 | -1167284 | +96 | -5488 | -411374 | -1704122 |
| +431 | -119 | -6843 | -622285 | +251 | -200 | -8524 | -18470 | +265 | -731 | -31086 | -39950 | +685 | -1850 | -73064 | -1292810 | +121 | -3587 | -381999 | -1870663 |
| +509 | +2173 | -5023 | -586975 | +302 | +13 | -7079 | -18590 | +317 | +159 | -25587 | -39909 | +838 | -819 | -64683 | -1358136 | +147 | -1431 | -335979 | -1950453 |
| +590 | +4730 | -2656 | -497552 | +355 | +252 | -5118 | -17397 | +371 | +1155 | -18207 | -37059 | +997 | +342 | -52758 | -1346931 | +174 | +990 | -271494 | -1920798 |
| +674 | +7562 | +312 | -343064 | +410 | +518 | -2588 | -14602 | +427 | +2261 | -8757 | -30799 | +1162 | +1638 | -36918 | -1241226 | +202 | +3686 | -186654 | -1756778 |
| +761 | +10679 | +3937 | -111569 | +467 | +812 | +566 | -9889 | +485 | +3481 | +2959 | -20473 | +1333 | +3074 | -16778 | -1021351 | +231 | +6667 | -79499 | -1431163 |
| +851 | +14091 | +8277 | +209901 | +526 | +1135 | +4401 | -2914 | +545 | +4819 | +17144 | -5368 | +1510 | +4655 | +8061 | -665872 | +261 | +9943 | +52001 | -914329 |
| +944 | +17808 | +13392 | +635376 | +587 | +1488 | +8976 | +6696 | +607 | +6279 | +34008 | +15288 | +1693 | +6386 | +38012 | -151528 | +292 | +13524 | +209946 | -174174 |
| +1040 | +21840 | +19344 | +1179984 | +650 | +1872 | +14352 | +19344 | +671 | +7865 | +53768 | +42328 | +1882 | +8272 | +73502 | +546832 | +324 | +17420 | +396506 | +823966 |
| | | | | +715 | +2288 | +20592 | +35464 | +737 | +9581 | +76648 | +76648 | +2077 | +10318 | +114972 | +1456312 | +357 | +21641 | +613921 | +2117401 |
| | | | | | | | | | | | | +2278 | +12529 | +162877 | +2606032 | +391 | +26197 | +864501 | +3746171 |
| 15,636,192 | | 3,716,201,632 | | | 59,938,736 | | 11,993,196,176 | | 1,063,912,564 | | 56,484,085,216 | | 1,841,387,130 | | 65,832,201,319,248 | | 8,146,743,060 | | 137,150,419,415,100 |
| | 5,394,486,240 | | 13,168.529,401,248 | 7,492,342 | | 4,255,650,256 | | 8,068,676 | | 59,579,103,584 | | 78,119,454 | | 271,829,660,102 | | 2,331,924 | | 7,736,690,325,980 | |
| $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{1}{12}$ | $\frac{3}{10}$ | 1 | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{120}$ | 1 | $\frac{2}{3}$ | $\frac{7}{24}$ | $\frac{1}{60}$ | 3 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{21}{40}$ | $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{35}{12}$ | $\frac{7}{10}$ |

| 71 | | | | 72 | | | | 73 | | | | 74 | | | | 75 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ | $\xi'_2$ | $\xi'_3$ | $\xi'_4$ | $\xi'_5$ |
| −420 | 0 | +79254 | 0 | −1295 | −259 | +83657 | +59755 | −444 | 0 | +12654 | 0 | −228 | −684 | +13338 | +22230 | −1406 | 0 | +493506 | 0 |
| −419 | −629 | +78625 | +131461 | −1289 | −775 | +82365 | +177973 | −443 | −133 | +12559 | +20995 | −227 | −2047 | +13143 | +66235 | −1403 | −702 | +489996 | +35100 |
| −416 | −1253 | +76745 | +259996 | −1277 | −1285 | +79795 | +292333 | −440 | −265 | +12275 | +41548 | −225 | −3395 | +12755 | +108881 | −1394 | −1399 | +479501 | +69500 |
| −411 | −1867 | +73635 | +382721 | −1259 | −1785 | +75975 | +400323 | −435 | −395 | +11805 | +61223 | −222 | −4718 | +12178 | +149282 | −1379 | −2086 | +462126 | +102509 |
| −404 | −2466 | +69330 | +496836 | −1235 | −2271 | +70947 | +499521 | −428 | −522 | +11154 | +79596 | −218 | −6006 | +11418 | +186582 | −1358 | −2758 | +438046 | +133454 |
| −395 | −3045 | +63879 | +599667 | −1205 | −2739 | +64767 | +587631 | −419 | −645 | +10329 | +96261 | −213 | −7249 | +10483 | +219967 | −1331 | −3410 | +407506 | +161689 |
| −384 | −3599 | +57345 | +688708 | −1169 | −3185 | +57505 | +662519 | −408 | −763 | +9339 | +110836 | −207 | −8437 | +9383 | +248677 | −1298 | −4037 | +370821 | +186604 |
| −371 | −4123 | +49805 | +761663 | −1127 | −3605 | +49245 | +722249 | −395 | −875 | +8195 | +122969 | −200 | −9560 | +8130 | +272018 | −1259 | −4634 | +328376 | +207634 |
| −356 | −4612 | +41350 | +816488 | −1079 | −3995 | +40085 | +765119 | −380 | −980 | +6910 | +132344 | −192 | −10608 | +6738 | +289374 | −1214 | −5196 | +280626 | +224268 |
| −339 | −5061 | +32085 | +851433 | −1025 | −4351 | +30137 | +789697 | −363 | −1077 | +5499 | +138687 | −183 | −11571 | +5223 | +300219 | −1163 | −5718 | +228096 | +236058 |
| −320 | −5465 | +22129 | +865084 | −965 | −4669 | +19527 | +794857 | −344 | −1165 | +3979 | +141772 | −173 | −12439 | +3603 | +304129 | −1106 | −6195 | +171381 | +242628 |
| −299 | −5819 | +11615 | +856405 | −899 | −4945 | +8395 | +779815 | −323 | −1243 | +2369 | +141427 | −162 | −13202 | +1898 | +300794 | −1043 | −6622 | +111146 | +243683 |
| −276 | −6118 | +690 | +824780 | −827 | −5175 | −3105 | +744165 | −300 | −1310 | +690 | +137540 | −150 | −13850 | +130 | +290030 | −974 | −6994 | +48126 | +239018 |
| −251 | −6357 | −10485 | +770055 | −749 | −5355 | −14805 | +687915 | −275 | −1365 | −1035 | +130065 | −137 | −14373 | −1677 | +271791 | −899 | −7306 | −16874 | +228527 |
| −224 | −6531 | −21735 | +692580 | −665 | −5481 | −26523 | +611523 | −248 | −1407 | −2781 | +119028 | −123 | −14761 | −3497 | +246181 | −818 | −7553 | −82979 | +212212 |
| −195 | −6635 | −32871 | +593251 | −575 | −5549 | −38063 | +515933 | −219 | −1435 | −4521 | +104533 | −108 | −15004 | −5302 | +213466 | −731 | −7730 | −149244 | +190192 |
| −164 | −6664 | −43690 | +473552 | −479 | −5555 | −49215 | +402611 | −188 | −1448 | −6226 | +86768 | −92 | −15092 | −7062 | +174086 | −638 | −7832 | −214654 | +162712 |
| −131 | −6613 | −53975 | +335597 | −377 | −5495 | −59755 | +273581 | −155 | −1445 | −7865 | +66011 | −75 | −15015 | −8745 | +128667 | −539 | −7854 | −278124 | +130152 |
| −96 | −6477 | −63495 | +182172 | −269 | −5365 | −69445 | +131461 | −120 | −1425 | −9405 | +42636 | −57 | −14763 | −10317 | +78033 | −434 | −7791 | −338499 | +93036 |
| −59 | −6251 | −72005 | +16777 | −155 | −5161 | −78033 | −20501 | −83 | −1387 | −10811 | +17119 | −38 | −14326 | −11742 | +23218 | −323 | −7638 | −394554 | +52041 |
| −20 | −5930 | −79246 | −156332 | −35 | −4879 | −85253 | −178391 | −44 | −1330 | −12046 | −9956 | −18 | −13694 | −12982 | −34522 | −206 | −7390 | −444994 | +8006 |
| +21 | −5509 | −84945 | −332101 | +91 | −4515 | −90825 | −337593 | −3 | −1253 | −13071 | −37891 | +3 | −12857 | −13997 | −93697 | −83 | −7042 | −488454 | −38059 |
| +64 | −4983 | −88815 | −504636 | +223 | −4065 | −94455 | −492753 | +40 | −1155 | −13845 | −65868 | +25 | −11805 | −14745 | −152571 | +46 | −6589 | −523499 | −84964 |
| +109 | −4347 | −90555 | −667161 | +361 | −3525 | −95835 | −637743 | +85 | −1035 | −14325 | −92943 | +48 | −10528 | −15182 | −209150 | +181 | −6026 | −548624 | −131330 |
| +156 | −3596 | −89850 | −811976 | +505 | −2891 | −94643 | −765625 | +132 | −892 | −14466 | −118040 | +72 | −9016 | −15262 | −261170 | +322 | −5348 | −562254 | −175580 |
| +205 | −2725 | −86371 | −930415 | +655 | −2159 | −90543 | −868615 | +181 | −725 | −14221 | −139945 | +97 | −7259 | −14937 | −306085 | +469 | −4550 | −562744 | −215930 |
| +256 | −1729 | −79775 | −1012804 | +811 | −1325 | −83185 | −938047 | +232 | −533 | −13541 | −157300 | +123 | −5247 | −14157 | −341055 | +622 | −3627 | −548379 | −250380 |
| +309 | −603 | −69705 | −1048419 | +973 | −385 | −72205 | −964337 | +285 | −315 | −12375 | −168597 | +150 | −2970 | −12870 | −362934 | +781 | −2574 | −517374 | −276705 |
| +364 | +658 | −55790 | −1025444 | +1141 | +665 | −57225 | −936947 | +340 | −70 | −10670 | −172172 | +178 | −418 | −11022 | −368258 | +946 | −1386 | −467874 | −292446 |
| +421 | +2059 | −37645 | −930929 | +1315 | +1829 | −37853 | −844349 | +397 | +203 | −8371 | −166199 | +207 | +2419 | −8557 | −353233 | +1117 | −58 | −397954 | −294901 |
| +480 | +3605 | −14871 | −750748 | +1495 | +3111 | −13683 | −673989 | +456 | +505 | −5421 | −148684 | +237 | +5551 | −5417 | −312723 | +1294 | +1415 | −305619 | −281116 |
| +541 | +5301 | +12945 | −469557 | +1681 | +4515 | +15705 | −412251 | +517 | +837 | −1761 | −117459 | +268 | +8988 | −1542 | −245238 | +1477 | +3038 | −188804 | −247876 |
| +604 | +7152 | +46230 | −70752 | +1873 | +6045 | +50745 | −44421 | +580 | +1200 | +2670 | −70176 | +300 | +12740 | +3130 | −142922 | +1666 | +4816 | −45374 | −191696 |
| +669 | +9163 | +85425 | +463573 | +2071 | +7705 | +91885 | +445349 | +645 | +1595 | +7935 | −4301 | +333 | +16817 | +8663 | −1541 | +1861 | +6754 | +126876 | −108812 |
| +736 | +11339 | +130985 | +1152668 | +2275 | +9499 | +139587 | +1074077 | +712 | +2023 | +14099 | +82892 | +367 | +21229 | +15123 | +184529 | +2062 | +8857 | +330221 | +4828 |
| +805 | +13685 | +183379 | +2017169 | +2485 | +11431 | +194327 | +1859987 | +781 | +2485 | +21229 | +194327 | +402 | +25986 | +22578 | +421314 | +2269 | +11130 | +567006 | +153573 |
| | | | | | | | | +852 | +2982 | +29394 | +333132 | +438 | +31098 | +31098 | +715254 | +2482 | +13578 | +839646 | +342078 |
| | | | | | | | | | | | | | | | | +2701 | +16206 | +1150626 | +575313 |
| | 2,249,473,830 | | 40,090,122,598,260 | | 1,587,863,880 | | 34,362,962,227,080 | | 109,309,470 | | 1,111,240,072,020 | | 12,024,041,700 | | 5,163,997,981,740 | | 3,302,377,650 | | 3,367,824,770,700 |
| 10,013,556 | | 351,667,742,090 | | 96,652,584 | | 398,906,692,520 | | 11,506,260 | | 9,218,431,970 | | 3,079,140 | | 10,420,836,140 | | 118,546,890 | | 14,406,805,963,550 | |
| 1 | $\frac{5}{6}$ | $\frac{7}{12}$ | $\frac{7}{20}$ | 3 | $\frac{2}{3}$ | $\frac{7}{12}$ | $\frac{3}{10}$ | 1 | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{20}$ | $\frac{1}{2}$ | $\frac{5}{3}$ | $\frac{1}{12}$ | $\frac{1}{10}$ | 3 | $\frac{5}{6}$ | $\frac{35}{12}$ | $\frac{3}{40}$ |

## TABLE XXIV. CALCULATION OF INTEGRALS FROM EQUALLY SPACED ORDINATES

Each panel over which the integral is required is divided by calculated ordinates into a number of sub-panels, usually an even number, and often a power of 2. The formula may involve only internal and terminal ordinates, or also adjacent external ordinates. Ordinates equidistant from the centre have equal coefficients. The central coefficient is the last given.

| Number of sub-panels. | No external ordinates. Coefficients. | Divisor. | One external ordinate. Coefficients. | Divisor. |
|---|---|---|---|---|
| 2 | 1, 4 | 6 | −1, 34, 114 | 180 |
| 4 | 7, 32, 12 | 90 | −4, 171, 612, 332 | 1890 |
| 6 | 41, 216, 27, 272 | 840 | −9, 482, 1908, 774, 2090 | 8400 |

Equally we may use the central ordinate and its even differences with the following coefficients:

| Number of sub-panels. | $\delta^0$ | $\delta^2$ | $\delta^4$ | $\delta^6$ | $\delta^8$ | $\delta^{10}$ |
|---|---|---|---|---|---|---|
| 2 | 1 | 0·1$\dot{6}$ | −·00$\dot{5}$ | ·0006$\dot{6}$ 137$\dot{5}$ | −·00010 141 | ·00001 76 |
| 4 | 1 | 0·6$\dot{6}$ | ·07$\dot{7}$ | −·002$\dot{1}$1 640$\dot{2}$ | ·00022 928 | −·00003 31 |
| 6 | 1 | 1·50 | ·550 | ·048$\dot{8}$0 952$\dot{3}$ | −·00107 143 | ·00010 28 |
| 8 | 1 | 2·6$\dot{6}$ | 1·91$\dot{1}$ | ·486$\dot{7}$7 248$\dot{6}$ | ·03488 536 | −·00063 28 |
| 10 | 1 | 4·1$\dot{6}$ | 4·86$\dot{1}$ | 2·278$\dot{4}$3 915$\dot{3}$ | ·44587 743 | ·02683 41 |

When the number of sub-panels is large, Gregory's formula, in terms of terminal differences, is valuable:

$$\int_0^r f(x)dx-(\tfrac{1}{2}f_0+f_1+f_2\ \cdots\ +f_{r-1}+\tfrac{1}{2}f_r)$$

$$=-a_1(\Delta f_{r-1}-\Delta f_0)-a_2(\Delta^2 f_{r-2}+\Delta^2 f_0)-a_3(\Delta^3 f_{r-3}-\Delta^3 f_0)-a_4(\Delta^4 f_{r-4}+\Delta^4 f_0)\ \cdots$$

where $a_r$ is the coefficient of $x^r$ in the expansion of $-1/\log(1-x)$.

### Values of $a_1$ to $a_{16}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | ·0833$\dot{3}$ | 5 | ·01426 91799 | 9 | ·00678 585 | 13 | ·00421 5 |
| 2 | ·0416$\dot{6}$ | 6 | ·01136 73942 | 10 | ·00592 406 | 14 | ·00382 7 |
| 3 | ·0263$\dot{8}$ | 7 | ·00935 6537 | 11 | ·00523 67 | 15 | ·00350 |
| 4 | ·01875 | 8 | ·00789 2554 | 12 | ·00467 75 | 16 | ·00321 |

## TABLE XXV. LOGARITHMS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | ·00000 | 00432 | 00860 | 01284 | 01703 | — | — | — | — | — | 43 | 85 | 128 | 170 | 213 |
| | — | — | — | — | — | 02119 | 02531 | 02938 | 03342 | 03743 | 41 | 81 | 122 | 162 | 203 |
| 11 | ·04139 | 04532 | 04922 | 05308 | 05690 | — | — | — | — | — | 39 | 78 | 116 | 155 | 194 |
| | — | — | — | — | — | 06070 | 06446 | 06819 | 07188 | 07555 | 37 | 74 | 111 | 148 | 186 |
| 12 | ·07918 | 08279 | 08636 | 08991 | 09342 | — | — | — | — | — | 36 | 71 | 107 | 142 | 178 |
| | — | — | — | — | — | 09691 | 10037 | 10380 | 10721 | 11059 | 34 | 68 | 103 | 137 | 171 |
| 13 | ·11394 | 11727 | 12057 | 12385 | 12710 | — | — | — | — | — | 33 | 66 | 99 | 132 | 164 |
| | — | — | — | — | — | 13033 | 13354 | 13672 | 13988 | 14301 | 32 | 63 | 95 | 127 | 158 |
| 14 | ·14613 | 14922 | 15229 | 15534 | 15836 | — | — | — | — | — | 31 | 61 | 92 | 122 | 153 |
| | — | — | — | — | — | 16137 | 16435 | 16732 | 17026 | 17319 | 30 | 59 | 89 | 118 | 148 |
| 15 | ·17609 | 17898 | 18184 | 18469 | 18752 | — | — | — | — | — | 29 | 57 | 86 | 114 | 143 |
| | — | — | — | — | — | 19033 | 19312 | 19590 | 19866 | 20140 | 28 | 55 | 83 | 111 | 138 |
| 16 | ·20412 | 20683 | 20952 | 21219 | 21484 | — | — | — | — | — | 27 | 54 | 80 | 107 | 134 |
| | — | — | — | — | — | 21748 | 22011 | 22272 | 22531 | 22789 | 26 | 52 | 78 | 104 | 130 |
| 17 | ·23045 | 23300 | 23553 | 23805 | 24055 | — | — | — | — | — | 25 | 50 | 76 | 101 | 126 |
| | — | — | — | — | — | 24304 | 24551 | 24797 | 25042 | 25285 | 25 | 49 | 74 | 98 | 123 |
| 18 | ·25527 | 25768 | 26007 | 26245 | 26482 | — | — | — | — | — | 24 | 48 | 72 | 96 | 119 |
| | — | — | — | — | — | 26717 | 26951 | 27184 | 27416 | 27646 | 23 | 46 | 70 | 93 | 116 |
| 19 | ·27875 | 28103 | 28330 | 28556 | 28780 | — | — | — | — | — | 23 | 45 | 69 | 90 | 113 |
| | — | — | — | — | — | 29003 | 29226 | 29447 | 29667 | 29885 | 22 | 44 | 66 | 88 | 110 |
| 20 | ·30103 | 30320 | 30535 | 30750 | 30963 | 31175 | 31387 | 31597 | 31806 | 32015 | 21 | 42 | 64 | 85 | 106 |
| 21 | ·32222 | 32428 | 32634 | 32838 | 33041 | 33244 | 33445 | 33646 | 33846 | 34044 | 20 | 40 | 61 | 81 | 101 |
| 22 | ·34242 | 34439 | 34635 | 34830 | 35025 | 35218 | 35411 | 35603 | 35793 | 35984 | 19 | 39 | 58 | 77 | 97 |
| 23 | ·36173 | 36361 | 36549 | 36736 | 36922 | 37107 | 37291 | 37475 | 37658 | 37840 | 19 | 37 | 56 | 74 | 93 |
| 24 | ·38021 | 38202 | 38382 | 38561 | 38739 | 38917 | 39094 | 39270 | 39445 | 39620 | 18 | 36 | 53 | 71 | 89 |
| 25 | ·39794 | 39967 | 40140 | 40312 | 40483 | 40654 | 40824 | 40993 | 41162 | 41330 | 17 | 34 | 51 | 68 | 85 |
| 26 | ·41497 | 41664 | 41830 | 41996 | 42160 | 42325 | 42488 | 42651 | 42813 | 42975 | 16 | 33 | 49 | 66 | 82 |
| 27 | ·43136 | 43297 | 43457 | 43616 | 43775 | 43933 | 44091 | 44248 | 44404 | 44560 | 16 | 32 | 47 | 63 | 79 |
| 28 | ·44716 | 44871 | 45025 | 45179 | 45332 | 45484 | 45637 | 45788 | 45939 | 46090 | 15 | 31 | 46 | 61 | 76 |
| 29 | ·46240 | 46389 | 46538 | 46687 | 46835 | 46982 | 47129 | 47276 | 47422 | 47567 | 15 | 29 | 44 | 59 | 74 |
| 30 | ·47712 | 47857 | 48001 | 48144 | 48287 | 48430 | 48572 | 48714 | 48855 | 48996 | 14 | 29 | 43 | 57 | 71 |
| 31 | ·49136 | 49276 | 49415 | 49554 | 49693 | 49831 | 49969 | 50106 | 50243 | 50379 | 14 | 28 | 41 | 55 | 69 |
| 32 | ·50515 | 50651 | 50786 | 50920 | 51055 | 51188 | 51322 | 51455 | 51587 | 51720 | 13 | 27 | 40 | 54 | 67 |
| 33 | ·51851 | 51983 | 52114 | 52244 | 52375 | 52504 | 52634 | 52763 | 52892 | 53020 | 13 | 26 | 39 | 52 | 65 |
| 34 | ·53148 | 53275 | 53403 | 53529 | 53656 | 53782 | 53908 | 54033 | 54158 | 54283 | 13 | 25 | 38 | 50 | 63 |
| 35 | ·54407 | 54531 | 54654 | 54777 | 54900 | 55023 | 55145 | 55267 | 55388 | 55509 | 12 | 24 | 37 | 49 | 61 |
| 36 | ·55630 | 55751 | 55871 | 55991 | 56110 | 56229 | 56348 | 56467 | 56585 | 56703 | 12 | 24 | 36 | 48 | 60 |
| 37 | ·56820 | 56937 | 57054 | 57171 | 57287 | 57403 | 57519 | 57634 | 57749 | 57864 | 12 | 23 | 35 | 46 | 58 |
| 38 | ·57978 | 58092 | 58206 | 58320 | 58433 | 58546 | 58659 | 58771 | 58883 | 58995 | 11 | 23 | 34 | 45 | 56 |
| 39 | ·59106 | 59218 | 59329 | 59439 | 59550 | 59660 | 59770 | 59879 | 59988 | 60097 | 11 | 22 | 33 | 44 | 55 |
| 40 | ·60206 | 60314 | 60423 | 60531 | 60638 | 60746 | 60853 | 60959 | 61066 | 61172 | 11 | 21 | 32 | 43 | 54 |
| 41 | ·61278 | 61384 | 61490 | 61595 | 61700 | 61805 | 61909 | 62014 | 62118 | 62221 | 10 | 21 | 31 | 42 | 52 |
| 42 | ·62325 | 62428 | 62531 | 62634 | 62737 | 62839 | 62941 | 63043 | 63144 | 63246 | 10 | 20 | 31 | 41 | 51 |
| 43 | ·63347 | 63448 | 63548 | 63649 | 63749 | 63849 | 63949 | 64048 | 64147 | 64246 | 10 | 20 | 30 | 40 | 50 |
| 44 | ·64345 | 64444 | 64542 | 64640 | 64738 | 64836 | 64933 | 65031 | 65128 | 65225 | 10 | 20 | 29 | 39 | 49 |
| 45 | ·65321 | 65418 | 65514 | 65610 | 65706 | 65801 | 65896 | 65992 | 66087 | 66181 | 10 | 19 | 29 | 38 | 48 |
| 46 | ·66276 | 66370 | 66464 | 66558 | 66652 | 66745 | 66839 | 66932 | 67025 | 67117 | 9 | 19 | 28 | 37 | 47 |
| 47 | ·67210 | 67302 | 67394 | 67486 | 67578 | 67669 | 67761 | 67852 | 67943 | 68034 | 9 | 18 | 27 | 37 | 46 |
| 48 | ·68124 | 68215 | 68305 | 68395 | 68485 | 68574 | 68664 | 68753 | 68842 | 68931 | 9 | 18 | 27 | 36 | 45 |
| 49 | ·69020 | 69108 | 69197 | 69285 | 69373 | 69461 | 69548 | 69636 | 69723 | 69810 | 9 | 18 | 26 | 35 | 44 |

## TABLE XXV. LOGARITHMS—*continued*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | ·69897 | 69984 | 70070 | 70157 | 70243 | 70329 | 70415 | 70501 | 70586 | 70672 | 9 | 17 | 26 | 34 | 43 |
| 51 | ·70757 | 70842 | 70927 | 71012 | 71096 | 71181 | 71265 | 71349 | 71433 | 71517 | 8 | 17 | 25 | 34 | 42 |
| 52 | ·71600 | 71684 | 71767 | 71850 | 71933 | 72016 | 72099 | 72181 | 72263 | 72346 | 8 | 17 | 25 | 33 | 41 |
| 53 | ·72428 | 72509 | 72591 | 72673 | 72754 | 72835 | 72916 | 72997 | 73078 | 73159 | 8 | 16 | 24 | 32 | 41 |
| 54 | ·73239 | 73320 | 73400 | 73480 | 73560 | 73640 | 73719 | 73799 | 73878 | 73957 | 8 | 16 | 24 | 32 | 40 |
| 55 | ·74036 | 74115 | 74194 | 74273 | 74351 | 74429 | 74507 | 74586 | 74663 | 74741 | 8 | 16 | 23 | 31 | 39 |
| 56 | ·74819 | 74896 | 74974 | 75051 | 75128 | 75205 | 75282 | 75358 | 75435 | 75511 | 8 | 15 | 23 | 31 | 38 |
| 57 | ·75587 | 75664 | 75740 | 75815 | 75891 | 75967 | 76042 | 76118 | 76193 | 76268 | 8 | 15 | 23 | 30 | 38 |
| 58 | ·76343 | 76418 | 76492 | 76567 | 76641 | 76716 | 76790 | 76864 | 76938 | 77012 | 7 | 15 | 22 | 30 | 37 |
| 59 | ·77085 | 77159 | 77232 | 77305 | 77379 | 77452 | 77525 | 77597 | 77670 | 77743 | 7 | 15 | 22 | 29 | 37 |
| 60 | ·77815 | 77887 | 77960 | 78032 | 78104 | 78176 | 78247 | 78319 | 78390 | 78462 | 7 | 14 | 22 | 29 | 36 |
| 61 | ·78533 | 78604 | 78675 | 78746 | 78817 | 78888 | 78958 | 79029 | 79099 | 79169 | 7 | 14 | 21 | 28 | 35 |
| 62 | ·79239 | 79309 | 79379 | 79449 | 79518 | 79588 | 79657 | 79727 | 79796 | 79865 | 7 | 14 | 21 | 28 | 35 |
| 63 | ·79934 | 80003 | 80072 | 80140 | 80209 | 80277 | 80346 | 80414 | 80482 | 80550 | 7 | 14 | 21 | 27 | 34 |
| 64 | ·80618 | 80686 | 80754 | 80821 | 80889 | 80956 | 81023 | 81090 | 81158 | 81224 | 7 | 13 | 20 | 27 | 34 |
| 65 | ·81291 | 81358 | 81425 | 81491 | 81558 | 81624 | 81690 | 81757 | 81823 | 81889 | 7 | 13 | 20 | 27 | 33 |
| 66 | ·81954 | 82020 | 82086 | 82151 | 82217 | 82282 | 82347 | 82413 | 82478 | 82543 | 7 | 13 | 20 | 26 | 33 |
| 67 | ·82607 | 82672 | 82737 | 82802 | 82866 | 82930 | 82995 | 83059 | 83123 | 83187 | 6 | 13 | 19 | 26 | 32 |
| 68 | ·83251 | 83315 | 83378 | 83442 | 83506 | 83569 | 83632 | 83696 | 83759 | 83822 | 6 | 13 | 19 | 25 | 32 |
| 69 | ·83885 | 83948 | 84011 | 84073 | 84136 | 84198 | 84261 | 84323 | 84386 | 84448 | 6 | 13 | 19 | 25 | 31 |
| 70 | ·84510 | 84572 | 84634 | 84696 | 84757 | 84819 | 84880 | 84942 | 85003 | 85065 | 6 | 12 | 19 | 25 | 31 |
| 71 | ·85126 | 85187 | 85248 | 85309 | 85370 | 85431 | 85491 | 85552 | 85612 | 85673 | 6 | 12 | 18 | 24 | 30 |
| 72 | ·85733 | 85794 | 85854 | 85914 | 85974 | 86034 | 86094 | 86153 | 86213 | 86273 | 6 | 12 | 18 | 24 | 30 |
| 73 | ·86332 | 86392 | 86451 | 86510 | 86570 | 86629 | 86688 | 86747 | 86806 | 86864 | 6 | 12 | 18 | 24 | 30 |
| 74 | ·86923 | 86982 | 87040 | 87099 | 87157 | 87216 | 87274 | 87332 | 87390 | 87448 | 6 | 12 | 17 | 23 | 29 |
| 75 | ·87506 | 87564 | 87622 | 87679 | 87737 | 87795 | 87852 | 87910 | 87967 | 88024 | 6 | 12 | 17 | 23 | 29 |
| 76 | ·88081 | 88138 | 88195 | 88252 | 88309 | 88366 | 88423 | 88480 | 88536 | 88593 | 6 | 11 | 17 | 23 | 28 |
| 77 | ·88649 | 88705 | 88762 | 88818 | 88874 | 88930 | 88986 | 89042 | 89098 | 89154 | 6 | 11 | 17 | 22 | 28 |
| 78 | ·89209 | 89265 | 89321 | 89376 | 89432 | 89487 | 89542 | 89597 | 89653 | 89708 | 6 | 11 | 17 | 22 | 28 |
| 79 | ·89763 | 89818 | 89873 | 89927 | 89982 | 90037 | 90091 | 90146 | 90200 | 90255 | 5 | 11 | 16 | 22 | 27 |
| 80 | ·90309 | 90363 | 90417 | 90472 | 90526 | 90580 | 90634 | 90687 | 90741 | 90795 | 5 | 11 | 16 | 22 | 27 |
| 81 | ·90849 | 90902 | 90956 | 91009 | 91062 | 91116 | 91169 | 91222 | 91275 | 91328 | 5 | 11 | 16 | 21 | 27 |
| 82 | ·91381 | 91434 | 91487 | 91540 | 91593 | 91645 | 91698 | 91751 | 91803 | 91855 | 5 | 11 | 16 | 21 | 26 |
| 83 | ·91908 | 91960 | 92012 | 92065 | 92117 | 92169 | 92221 | 92273 | 92324 | 92376 | 5 | 10 | 16 | 21 | 26 |
| 84 | ·92428 | 92480 | 92531 | 92583 | 92634 | 92686 | 92737 | 92788 | 92840 | 92891 | 5 | 10 | 15 | 21 | 26 |
| 85 | ·92942 | 92993 | 93044 | 93095 | 93146 | 93197 | 93247 | 93298 | 93349 | 93399 | 5 | 10 | 15 | 20 | 25 |
| 86 | ·93450 | 93500 | 93551 | 93601 | 93651 | 93702 | 93752 | 93802 | 93852 | 93902 | 5 | 10 | 15 | 20 | 25 |
| 87 | ·93952 | 94002 | 94052 | 94101 | 94151 | 94201 | 94250 | 94300 | 94349 | 94399 | 5 | 10 | 15 | 20 | 25 |
| 88 | ·94448 | 94498 | 94547 | 94596 | 94645 | 94694 | 94743 | 94792 | 94841 | 94890 | 5 | 10 | 15 | 20 | 25 |
| 89 | ·94939 | 94988 | 95036 | 95085 | 95134 | 95182 | 95231 | 95279 | 95328 | 95376 | 5 | 10 | 15 | 19 | 24 |
| 90 | ·95424 | 95472 | 95521 | 95569 | 95617 | 95665 | 95713 | 95761 | 95809 | 95856 | 5 | 10 | 14 | 19 | 24 |
| 91 | ·95904 | 95952 | 95999 | 96047 | 96095 | 96142 | 96190 | 96237 | 96284 | 96332 | 5 | 10 | 14 | 19 | 24 |
| 92 | ·96379 | 96426 | 96473 | 96520 | 96567 | 96614 | 96661 | 96708 | 96755 | 96802 | 5 | 9 | 14 | 19 | 24 |
| 93 | ·96848 | 96895 | 96942 | 96988 | 97035 | 97081 | 97128 | 97174 | 97220 | 97267 | 5 | 9 | 14 | 19 | 23 |
| 94 | ·97313 | 97359 | 97405 | 97451 | 97497 | 97543 | 97589 | 97635 | 97681 | 97727 | 5 | 9 | 14 | 18 | 23 |
| 95 | ·97772 | 97818 | 97864 | 97909 | 97955 | 98000 | 98046 | 98091 | 98137 | 98182 | 5 | 9 | 14 | 18 | 23 |
| 96 | ·98227 | 98272 | 98318 | 98363 | 98408 | 98453 | 98498 | 98543 | 98588 | 98632 | 4 | 9 | 14 | 18 | 22 |
| 97 | ·98677 | 98722 | 98767 | 98811 | 98856 | 98900 | 98945 | 98989 | 99034 | 99078 | 4 | 9 | 13 | 18 | 22 |
| 98 | ·99123 | 99167 | 99211 | 99255 | 99300 | 99344 | 99388 | 99432 | 99476 | 99520 | 4 | 9 | 13 | 18 | 22 |
| 99 | ·99564 | 99607 | 99651 | 99695 | 99739 | 99782 | 99826 | 99870 | 99913 | 99957 | 4 | 9 | 13 | 17 | 22 |

# TABLE XXVI. NATURAL LOGARITHMS

Negative Values, Decreasing

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·10 | 2·30259 | 29263 | 28278 | 27303 | 26336 | — | — | — | — | — | 98 | 196 | 294 | 392 | 490 |
|  | — | — | — | — | — | 25379 | 24432 | 23493 | 22562 | 21641 | 93 | 187 | 280 | 374 | 467 |
| ·11 | 2·20727 | 19823 | 18926 | 18037 | 17156 | — | — | — | — | — | 89 | 179 | 268 | 357 | 446 |
|  | — | — | — | — | — | 16282 | 15417 | 14558 | 13707 | 12863 | 85 | 171 | 256 | 342 | 427 |
| ·12 | 2·12026 | 11196 | 10373 | 09557 | 08747 | — | — | — | — | — | 82 | 164 | 246 | 328 | 410 |
|  | — | — | — | — | — | 07944 | 07147 | 06357 | 05573 | 04794 | 79 | 158 | 236 | 315 | 394 |
| ·13 | 2·04022 | 03256 | 02495 | 01741 | 00992 | — | — | — | — | — | 76 | 152 | 227 | 303 | 379 |
|  | — | — | — | — | — | 00248 | **99510** | **98777** | **98050** | **97328** | 73 | 146 | 219 | 292 | 365 |
| ·14 | 1·96611 | 95900 | 95193 | 94491 | 93794 | — | — | — | — | — | 70 | 141 | 211 | 282 | 352 |
|  | — | — | — | — | — | 93102 | 92415 | 91732 | 91054 | 90381 | 68 | 136 | 204 | 272 | 340 |
| ·15 | 1·89712 | 89048 | 88387 | 87732 | 87080 | — | — | — | — | — | 66 | 132 | 197 | 263 | 329 |
|  | — | — | — | — | — | 86433 | 85790 | 85151 | 84516 | 83885 | 64 | 127 | 191 | 255 | 318 |
| ·16 | 1·83258 | 82635 | 82016 | 81401 | 80789 | — | — | — | — | — | 62 | 123 | 185 | 247 | 309 |
|  | — | — | — | — | — | 80181 | 79577 | 78976 | 78379 | 77786 | 60 | 120 | 180 | 240 | 299 |
| ·17 | 1·77196 | 76609 | 76026 | 75446 | 74870 | — | — | — | — | — | 58 | 116 | 174 | 233 | 291 |
|  | — | — | — | — | — | 74297 | 73727 | 73161 | 72597 | 72037 | 56 | 113 | 170 | 226 | 282 |
| ·18 | 1·71480 | 70926 | 70375 | 69827 | 69282 | — | — | — | — | — | 55 | 110 | 165 | 220 | 275 |
|  | — | — | — | — | — | 68740 | 68201 | 67665 | 67131 | 66601 | 53 | 107 | 160 | 214 | 267 |
| ·19 | 1·66073 | 65548 | 65026 | 64507 | 63990 | — | — | — | — | — | 52 | 104 | 156 | 208 | 260 |
|  | — | — | — | — | — | 63476 | 62964 | 62455 | 61949 | 61445 | 51 | 102 | 152 | 203 | 254 |
| ·20 | 1·60944 | 60445 | 59949 | 59455 | 58964 | 58475 | 57988 | 57504 | 57022 | 56542 | 49 | 98 | 147 | 196 | 245 |
| ·21 | 1·56065 | 55590 | 55117 | 54646 | 54178 | 53712 | 53248 | 52786 | 52326 | 51868 | 47 | 93 | 140 | 187 | 233 |
| ·22 | 1·51413 | 50959 | 50508 | 50058 | 49611 | 49165 | 48722 | 48281 | 47841 | 47403 | 45 | 89 | 134 | 178 | 223 |
| ·23 | 1·46968 | 46534 | 46102 | 45672 | 45243 | 44817 | 44392 | 43970 | 43548 | 43129 | 43 | 85 | 128 | 171 | 213 |
| ·24 | 1·42712 | 42296 | 41882 | 41469 | 41059 | 40650 | 40242 | 39837 | 39433 | 39030 | 41 | 82 | 123 | 164 | 205 |
| ·25 | 1·38629 | 38230 | 37833 | 37437 | 37042 | 36649 | 36258 | 35868 | 35480 | 35093 | 39 | 79 | 118 | 157 | 196 |
| ·26 | 1·34707 | 34323 | 33941 | 33560 | 33181 | 32803 | 32426 | 32051 | 31677 | 31304 | 38 | 76 | 113 | 151 | 189 |
| ·27 | 1·30933 | 30564 | 30195 | 29828 | 29463 | 29098 | 28735 | 28374 | 28013 | 27654 | 36 | 73 | 109 | 146 | 182 |
| ·28 | 1·27297 | 26940 | 26585 | 26231 | 25878 | 25527 | 25176 | 24827 | 24479 | 24133 | 35 | 70 | 105 | 141 | 176 |
| ·29 | 1·23787 | 23443 | 23100 | 22758 | 22418 | 22078 | 21740 | 21402 | 21066 | 20731 | 34 | 68 | 102 | 136 | 170 |
| ·30 | 1·20397 | 20065 | 19733 | 19402 | 19073 | 18744 | 18417 | 18091 | 17766 | 17441 | 33 | 66 | 99 | 131 | 164 |
| ·31 | 1·17118 | 16796 | 16475 | 16155 | 15836 | 15518 | 15201 | 14885 | 14570 | 14256 | 32 | 64 | 95 | 127 | 159 |
| ·32 | 1·13943 | 13631 | 13320 | 13010 | 12701 | 12393 | 12086 | 11780 | 11474 | 11170 | 31 | 62 | 92 | 123 | 154 |
| ·33 | 1·10866 | 10564 | 10262 | 09961 | 09661 | 09362 | 09064 | 08767 | 08471 | 08176 | 30 | 60 | 90 | 120 | 149 |
| ·34 | 1·07881 | 07587 | 07294 | 07002 | 06711 | 06421 | 06132 | 05843 | 05555 | 05268 | 29 | 58 | 87 | 116 | 145 |
| ·35 | 1·04982 | 04697 | 04412 | 04129 | 03846 | 03564 | 03282 | 03002 | 02722 | 02443 | 28 | 56 | 85 | 113 | 141 |
| ·36 | 1·02165 | 01888 | 01611 | 01335 | 01060 | 00786 | 00512 | 00239 | **99967** | **99696** | 27 | 55 | 82 | 110 | 137 |
| ·37 | 0·99425 | 99155 | 98886 | 98618 | 98350 | 98083 | 97817 | 97551 | 97286 | 97022 | 27 | 53 | 80 | 107 | 134 |
| ·38 | 0·96758 | 96496 | 96233 | 95972 | 95711 | 95451 | 95192 | 94933 | 94675 | 94418 | 26 | 52 | 78 | 104 | 130 |
| ·39 | 0·94161 | 93905 | 93649 | 93395 | 93140 | 92887 | 92634 | 92382 | 92130 | 91879 | 25 | 51 | 76 | 101 | 127 |
| ·40 | 0·91629 | 91379 | 91130 | 90882 | 90634 | 90387 | 90140 | 89894 | 89649 | 89404 | 25 | 49 | 74 | 99 | 124 |
| ·41 | 0·89160 | 88916 | 88673 | 88431 | 88189 | 87948 | 87707 | 87467 | 87227 | 86988 | 24 | 48 | 72 | 96 | 121 |
| ·42 | 0·86750 | 86512 | 86275 | 86038 | 85802 | 85567 | 85332 | 85097 | 84863 | 84630 | 24 | 47 | 71 | 94 | 118 |
| ·43 | 0·84397 | 84165 | 83933 | 83702 | 83471 | 83241 | 83011 | 82782 | 82554 | 82326 | 23 | 46 | 69 | 92 | 115 |
| ·44 | 0·82098 | 81871 | 81645 | 81419 | 81193 | 80968 | 80744 | 80520 | 80296 | 80073 | 22 | 45 | 68 | 90 | 112 |
| ·45 | 0·79851 | 79629 | 79407 | 79186 | 78966 | 78746 | 78526 | 78307 | 78089 | 77871 | 22 | 44 | 66 | 88 | 110 |
| ·46 | 0·77653 | 77436 | 77219 | 77003 | 76787 | 76572 | 76357 | 76143 | 75929 | 75715 | 22 | 43 | 65 | 86 | 108 |
| ·47 | 0·75502 | 75290 | 75078 | 74866 | 74655 | 74444 | 74234 | 74024 | 73814 | 73605 | 21 | 42 | 63 | 84 | 105 |
| ·48 | 0·73397 | 73189 | 72981 | 72774 | 72567 | 72361 | 72155 | 71949 | 71744 | 71539 | 21 | 41 | 62 | 83 | 103 |
| ·49 | 0·71335 | 71131 | 70928 | 70725 | 70522 | 70320 | 70118 | 69917 | 69716 | 69515 | 20 | 40 | 61 | 81 | 101 |

$\log_e 10 = 2{\cdot}30259$, $\log_e 100 = 4{\cdot}60517$, $\log_e 1000 = 6{\cdot}90776$

## TABLE XXVI. NATURAL LOGARITHMS—*continued*

Negative Values, Decreasing

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·50 | 0·69315 | 69115 | 68916 | 68717 | 68518 | 68320 | 68122 | 67924 | 67727 | 67531 | 20 | 40 | 59 | 79 | 99 |
| ·51 | 0·67334 | 67139 | 66943 | 66748 | 66553 | 66359 | 66165 | 65971 | 65778 | 65585 | 19 | 39 | 58 | 78 | 97 |
| ·52 | 0·65393 | 65201 | 65009 | 64817 | 64626 | 64436 | 64245 | 64055 | 63866 | 63677 | 19 | 38 | 57 | 76 | 95 |
| ·53 | 0·63488 | 63299 | 63111 | 62923 | 62736 | 62549 | 62362 | 62176 | 61990 | 61804 | 19 | 37 | 56 | 75 | 94 |
| ·54 | 0·61619 | 61434 | 61249 | 61065 | 60881 | 60697 | 60514 | 60331 | 60148 | 59966 | 18 | 37 | 55 | 73 | 92 |
| ·55 | 0·59784 | 59602 | 59421 | 59240 | 59059 | 58879 | 58699 | 58519 | 58340 | 58161 | 18 | 36 | 54 | 72 | 90 |
| ·56 | 0·57982 | 57803 | 57625 | 57448 | 57270 | 57093 | 56916 | 56740 | 56563 | 56387 | 18 | 35 | 53 | 71 | 89 |
| ·57 | 0·56212 | 56037 | 55862 | 55687 | 55513 | 55339 | 55165 | 54991 | 54818 | 54645 | 17 | 35 | 52 | 70 | 87 |
| ·58 | 0·54473 | 54300 | 54128 | 53957 | 53785 | 53614 | 53444 | 53273 | 53103 | 52933 | 17 | 34 | 51 | 68 | 86 |
| ·59 | 0·52763 | 52594 | 52425 | 52256 | 52088 | 51919 | 51751 | 51584 | 51416 | 51249 | 17 | 34 | 50 | 67 | 84 |
| ·60 | 0·51083 | 50916 | 50750 | 50584 | 50418 | 50253 | 50088 | 49923 | 49758 | 49594 | 17 | 33 | 50 | 66 | 83 |
| ·61 | 0·49430 | 49266 | 49102 | 48939 | 48776 | 48613 | 48451 | 48289 | 48127 | 47965 | 16 | 33 | 49 | 65 | 81 |
| ·62 | 0·47804 | 47642 | 47482 | 47321 | 47160 | 47000 | 46840 | 46681 | 46522 | 46362 | 16 | 32 | 48 | 64 | 80 |
| ·63 | 0·46204 | 46045 | 45887 | 45728 | 45571 | 45413 | 45256 | 45099 | 44942 | 44785 | 16 | 32 | 47 | 63 | 79 |
| ·64 | 0·44629 | 44473 | 44317 | 44161 | 44006 | 43850 | 43696 | 43541 | 43386 | 43232 | 16 | 31 | 47 | 62 | 78 |
| ·65 | 0·43078 | 42925 | 42771 | 42618 | 42465 | 42312 | 42159 | 42007 | 41855 | 41703 | 15 | 31 | 46 | 61 | 76 |
| ·66 | 0·41552 | 41400 | 41249 | 41098 | 40947 | 40797 | 40647 | 40497 | 40347 | 40197 | 15 | 30 | 45 | 60 | 75 |
| ·67 | 0·40048 | 39899 | 39750 | 39601 | 39453 | 39304 | 39156 | 39008 | 38861 | 38713 | 15 | 30 | 44 | 59 | 74 |
| ·68 | 0·38566 | 38419 | 38273 | 38126 | 37980 | 37834 | 37688 | 37542 | 37397 | 37251 | 15 | 29 | 44 | 58 | 73 |
| ·69 | 0·37106 | 36962 | 36817 | 36673 | 36528 | 36384 | 36241 | 36097 | 35954 | 35810 | 14 | 29 | 43 | 58 | 72 |
| ·70 | 0·35668 | 35525 | 35382 | 35240 | 35098 | 34956 | 34814 | 34672 | 34531 | 34390 | 14 | 28 | 43 | 57 | 71 |
| ·71 | 0·34249 | 34108 | 33968 | 33827 | 33687 | 33547 | 33408 | 33268 | 33129 | 32989 | 14 | 28 | 42 | 56 | 70 |
| ·72 | 0·32850 | 32712 | 32573 | 32435 | 32296 | 32158 | 32021 | 31883 | 31745 | 31608 | 14 | 28 | 41 | 55 | 69 |
| ·73 | 0·31471 | 31334 | 31197 | 31061 | 30925 | 30788 | 30653 | 30517 | 30381 | 30246 | 14 | 27 | 41 | 54 | 68 |
| ·74 | 0·30111 | 29975 | 29841 | 29706 | 29571 | 29437 | 29303 | 29169 | 29035 | 28902 | 13 | 27 | 40 | 54 | 67 |
| ·75 | 0·28768 | 28635 | 28502 | 28369 | 28236 | 28104 | 27971 | 27839 | 27707 | 27575 | 13 | 27 | 40 | 53 | 66 |
| ·76 | 0·27444 | 27312 | 27181 | 27050 | 26919 | 26788 | 26657 | 26527 | 26397 | 26266 | 13 | 26 | 39 | 52 | 65 |
| ·77 | 0·26136 | 26007 | 25877 | 25748 | 25618 | 25489 | 25360 | 25231 | 25103 | 24974 | 13 | 26 | 39 | 52 | 65 |
| ·78 | 0·24846 | 24718 | 24590 | 24462 | 24335 | 24207 | 24080 | 23953 | 23826 | 23699 | 13 | 25 | 38 | 51 | 64 |
| ·79 | 0·23572 | 23446 | 23319 | 23193 | 23067 | 22941 | 22816 | 22690 | 22565 | 22439 | 13 | 25 | 38 | 50 | 63 |
| ·80 | 0·22314 | 22189 | 22065 | 21940 | 21816 | 21691 | 21567 | 21443 | 21319 | 21196 | 12 | 25 | 37 | 50 | 62 |
| ·81 | 0·21072 | 20949 | 20825 | 20702 | 20579 | 20457 | 20334 | 20212 | 20089 | 19967 | 12 | 25 | 37 | 49 | 61 |
| ·82 | 0·19845 | 19723 | 19601 | 19480 | 19358 | 19237 | 19116 | 18995 | 18874 | 18754 | 12 | 24 | 36 | 49 | 61 |
| ·83 | 0·18633 | 18513 | 18392 | 18272 | 18152 | 18032 | 17913 | 17793 | 17674 | 17555 | 12 | 24 | 36 | 48 | 60 |
| ·84 | 0·17435 | 17316 | 17198 | 17079 | 16960 | 16842 | 16724 | 16605 | 16487 | 16370 | 12 | 24 | 36 | 47 | 59 |
| ·85 | 0·16252 | 16134 | 16017 | 15900 | 15782 | 15665 | 15548 | 15432 | 15315 | 15199 | 12 | 23 | 35 | 47 | 58 |
| ·86 | 0·15082 | 14966 | 14850 | 14734 | 14618 | 14503 | 14387 | 14272 | 14156 | 14041 | 12 | 23 | 35 | 46 | 58 |
| ·87 | 0·13926 | 13811 | 13697 | 13582 | 13467 | 13353 | 13239 | 13125 | 13011 | 12897 | 11 | 23 | 34 | 46 | 57 |
| ·88 | 0·12783 | 12670 | 12556 | 12443 | 12330 | 12217 | 12104 | 11991 | 11878 | 11766 | 11 | 23 | 34 | 45 | 56 |
| ·89 | 0·11653 | 11541 | 11429 | 11317 | 11205 | 11093 | 10981 | 10870 | 10759 | 10647 | 11 | 22 | 34 | 45 | 56 |
| ·90 | 0·10536 | 10425 | 10314 | 10203 | 10093 | 09982 | 09872 | 09761 | 09651 | 09541 | 11 | 22 | 33 | 44 | 55 |
| ·91 | 0·09431 | 09321 | 09212 | 09102 | 08992 | 08883 | 08774 | 08665 | 08556 | 08447 | 11 | 22 | 33 | 44 | 55 |
| ·92 | 0·08338 | 08230 | 08121 | 08013 | 07904 | 07796 | 07688 | 07580 | 07472 | 07365 | 11 | 22 | 32 | 43 | 54 |
| ·93 | 0·07257 | 07150 | 07042 | 06935 | 06828 | 06721 | 06614 | 06507 | 06401 | 06294 | 11 | 21 | 32 | 43 | 54 |
| ·94 | 0·06188 | 06081 | 05975 | 05869 | 05763 | 05657 | 05551 | 05446 | 05340 | 05235 | 11 | 21 | 32 | 42 | 53 |
| ·95 | 0·05129 | 05024 | 04919 | 04814 | 04709 | 04604 | 04500 | 04395 | 04291 | 04186 | 10 | 21 | 31 | 42 | 52 |
| ·96 | 0·04082 | 03978 | 03874 | 03770 | 03666 | 03563 | 03459 | 03356 | 03252 | 03149 | 10 | 21 | 31 | 41 | 52 |
| ·97 | 0·03046 | 02943 | 02840 | 02737 | 02634 | 02532 | 02429 | 02327 | 02225 | 02122 | 10 | 21 | 31 | 41 | 51 |
| ·98 | 0·02020 | 01918 | 01816 | 01715 | 01613 | 01511 | 01410 | 01309 | 01207 | 01106 | 10 | 20 | 30 | 41 | 51 |
| ·99 | 0·01005 | 00904 | 00803 | 00702 | 00602 | 00501 | 00401 | 00300 | 00200 | 00100 | 10 | 20 | 30 | 40 | 50 |

$\log_e 10{,}000 = 9{\cdot}21034$, $\log_e 100{,}000 = 11{\cdot}51293$, $\log_e 1{,}000{,}000 = 13{\cdot}81551$

## TABLE XXVI. NATURAL LOGARITHMS—*continued*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1·0 | 0·00000 | 00995 | 01980 | 02956 | 03922 | — | — | — | — | — | 98 | 196 | 294 | 392 | 490 |
| | — | — | — | — | — | 04879 | 05827 | 06766 | 07696 | 08618 | 93 | 187 | 280 | 374 | 467 |
| 1·1 | 0·09531 | 10436 | 11333 | 12222 | 13103 | — | — | — | — | — | 89 | 179 | 268 | 357 | 446 |
| | — | — | — | — | — | 13976 | 14842 | 15700 | 16551 | 17395 | 85 | 171 | 256 | 342 | 427 |
| 1·2 | 0·18232 | 19062 | 19885 | 20701 | 21511 | — | — | — | — | — | 82 | 164 | 246 | 328 | 410 |
| | — | — | — | — | — | 22314 | 23111 | 23902 | 24686 | 25464 | 79 | 158 | 236 | 315 | 394 |
| 1·3 | 0·26236 | 27003 | 27763 | 28518 | 29267 | — | — | — | — | — | 76 | 152 | 227 | 303 | 379 |
| | — | — | — | — | — | 30010 | 30748 | 31481 | 32208 | 32930 | 73 | 146 | 219 | 292 | 365 |
| 1·4 | 0·33647 | 34359 | 35066 | 35767 | 36464 | — | — | — | — | — | 70 | 141 | 211 | 282 | 352 |
| | — | — | — | — | — | 37156 | 37844 | 38526 | 39204 | 39878 | 68 | 136 | 204 | 272 | 340 |
| 1·5 | 0·40547 | 41211 | 41871 | 42527 | 43178 | — | — | — | — | — | 66 | 132 | 197 | 263 | 329 |
| | — | — | — | — | — | 43825 | 44469 | 45108 | 45742 | 46373 | 64 | 127 | 191 | 255 | 318 |
| 1·6 | 0·47000 | 47623 | 48243 | 48858 | 49470 | — | — | — | — | — | 62 | 124 | 185 | 247 | 309 |
| | — | — | — | — | — | 50078 | 50682 | 51282 | 51879 | 52473 | 60 | 120 | 180 | 240 | 299 |
| 1·7 | 0·53063 | 53649 | 54232 | 54812 | 55389 | — | — | — | — | — | 58 | 116 | 174 | 233 | 291 |
| | — | — | — | — | — | 55962 | 56531 | 57098 | 57661 | 58222 | 56 | 113 | 169 | 226 | 282 |
| 1·8 | 0·58779 | 59333 | 59884 | 60432 | 60977 | — | — | — | — | — | 55 | 110 | 165 | 220 | 275 |
| | — | — | — | — | — | 61519 | 62058 | 62594 | 63127 | 63658 | 53 | 107 | 160 | 214 | 267 |
| 1·9 | 0·64185 | 64710 | 65233 | 65752 | 66269 | — | — | — | — | — | 52 | 104 | 156 | 208 | 260 |
| | — | — | — | — | — | 66783 | 67294 | 67803 | 68310 | 68813 | 51 | 102 | 152 | 203 | 254 |
| 2·0 | 0·69315 | 69813 | 70310 | 70804 | 71295 | 71784 | 72271 | 72755 | 73237 | 73716 | 49 | 98 | 147 | 196 | 244 |
| 2·1 | 0·74194 | 74669 | 75142 | 75612 | 76081 | 76547 | 77011 | 77473 | 77932 | 78390 | 47 | 93 | 140 | 186 | 233 |
| 2·2 | 0·78846 | 79299 | 79751 | 80200 | 80648 | 81093 | 81536 | 81978 | 82418 | 82855 | 45 | 89 | 134 | 178 | 223 |
| 2·3 | 0·83291 | 83725 | 84157 | 84587 | 85015 | 85442 | 85866 | 86289 | 86710 | 87129 | 43 | 85 | 128 | 171 | 213 |
| 2·4 | 0·87547 | 87963 | 88377 | 88789 | 89200 | 89609 | 90016 | 90422 | 90826 | 91228 | 41 | 82 | 123 | 164 | 204 |
| 2·5 | 0·91629 | 92028 | 92426 | 92822 | 93216 | 93609 | 94001 | 94391 | 94779 | 95166 | 39 | 79 | 118 | 157 | 196 |
| 2·6 | 0·95551 | 95935 | 96317 | 96698 | 97078 | 97456 | 97833 | 98208 | 98582 | 98954 | 38 | 76 | 113 | 151 | 189 |
| 2·7 | 0·99325 | 99695 | **00063** | **00430** | **00796** | **01160** | **01523** | **01885** | **02245** | **02604** | 36 | 73 | 109 | 146 | 182 |
| 2·8 | 1·02962 | 03318 | 03674 | 04028 | 04380 | 04732 | 05082 | 05431 | 05779 | 06126 | 35 | 70 | 105 | 141 | 176 |
| 2·9 | 1·06471 | 06815 | 07158 | 07500 | 07841 | 08181 | 08519 | 08856 | 09192 | 09527 | 34 | 68 | 102 | 136 | 170 |
| 3·0 | 1·09861 | 10194 | 10526 | 10856 | 11186 | 11514 | 11841 | 12168 | 12493 | 12817 | 33 | 66 | 99 | 131 | 164 |
| 3·1 | 1·13140 | 13462 | 13783 | 14103 | 14422 | 14740 | 15057 | 15373 | 15688 | 16002 | 32 | 64 | 95 | 127 | 159 |
| 3·2 | 1·16315 | 16627 | 16938 | 17248 | 17557 | 17865 | 18173 | 18479 | 18784 | 19089 | 31 | 62 | 92 | 123 | 154 |
| 3·3 | 1·19392 | 19695 | 19996 | 20297 | 20597 | 20896 | 21194 | 21491 | 21788 | 22083 | 30 | 60 | 90 | 120 | 150 |
| 3·4 | 1·22378 | 22671 | 22964 | 23256 | 23547 | 23837 | 24127 | 24415 | 24703 | 24990 | 29 | 58 | 87 | 116 | 145 |
| 3·5 | 1·25276 | 25562 | 25846 | 26130 | 26413 | 26695 | 26976 | 27257 | 27536 | 27815 | 28 | 56 | 85 | 113 | 141 |
| 3·6 | 1·28093 | 28371 | 28647 | 28923 | 29198 | 29473 | 29746 | 30019 | 30291 | 30563 | 27 | 55 | 82 | 110 | 137 |
| 3·7 | 1·30833 | 31103 | 31372 | 31641 | 31909 | 32176 | 32442 | 32708 | 32972 | 33237 | 27 | 53 | 80 | 107 | 134 |
| 3·8 | 1·33500 | 33763 | 34025 | 34286 | 34547 | 34807 | 35067 | 35325 | 35584 | 35841 | 26 | 52 | 78 | 104 | 130 |
| 3·9 | 1·36098 | 36354 | 36609 | 36864 | 37118 | 37372 | 37624 | 37877 | 38128 | 38379 | 25 | 51 | 76 | 101 | 127 |
| 4·0 | 1·38629 | 38879 | 39128 | 39377 | 39624 | 39872 | 40118 | 40364 | 40610 | 40854 | 25 | 49 | 74 | 99 | 124 |
| 4·1 | 1·41099 | 41342 | 41585 | 41828 | 42070 | 42311 | 42552 | 42792 | 43031 | 43270 | 24 | 48 | 72 | 96 | 121 |
| 4·2 | 1·43508 | 43746 | 43984 | 44220 | 44456 | 44692 | 44927 | 45161 | 45395 | 45629 | 24 | 47 | 71 | 94 | 118 |
| 4·3 | 1·45862 | 46094 | 46326 | 46557 | 46787 | 47018 | 47247 | 47476 | 47705 | 47933 | 23 | 46 | 69 | 92 | 115 |
| 4·4 | 1·48160 | 48387 | 48614 | 48840 | 49065 | 49290 | 49515 | 49739 | 49962 | 50185 | 22 | 45 | 68 | 90 | 112 |
| 4·5 | 1·50408 | 50630 | 50851 | 51072 | 51293 | 51513 | 51732 | 51951 | 52170 | 52388 | 22 | 44 | 66 | 88 | 110 |
| 4·6 | 1·52606 | 52823 | 53039 | 53256 | 53471 | 53687 | 53902 | 54116 | 54330 | 54543 | 22 | 43 | 65 | 86 | 108 |
| 4·7 | 1·54756 | 54969 | 55181 | 55393 | 55604 | 55814 | 56025 | 56235 | 56444 | 56653 | 21 | 42 | 63 | 84 | 105 |
| 4·8 | 1·56862 | 57070 | 57277 | 57485 | 57691 | 57898 | 58104 | 58309 | 58515 | 58719 | 21 | 41 | 62 | 83 | 103 |
| 4·9 | 1·58924 | 59127 | 59331 | 59534 | 59737 | 59939 | 60141 | 60342 | 60543 | 60744 | 20 | 40 | 61 | 81 | 101 |

$\log_e 10 = 2{\cdot}30259$, $\log_e 100 = 4{\cdot}60517$, $\log_e 1000 = 6{\cdot}90776$

## TABLE XXVI. NATURAL LOGARITHMS—*continued*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5·0 | 1·60944 | 61144 | 61343 | 61542 | 61741 | 61939 | 62137 | 62334 | 62531 | 62728 | 20 | 40 | 59 | 79 | 99 |
| 5·1 | 1·62924 | 63120 | 63315 | 63511 | 63705 | 63900 | 64094 | 64287 | 64481 | 64673 | 19 | 39 | 58 | 78 | 97 |
| 5·2 | 1·64866 | 65058 | 65250 | 65441 | 65632 | 65823 | 66013 | 66203 | 66393 | 66582 | 19 | 38 | 57 | 76 | 95 |
| 5·3 | 1·66771 | 66959 | 67147 | 67335 | 67523 | 67710 | 67896 | 68083 | 68269 | 68455 | 19 | 37 | 56 | 75 | 94 |
| 5·4 | 1·68640 | 68825 | 69010 | 69194 | 69378 | 69562 | 69745 | 69928 | 70111 | 70293 | 18 | 37 | 55 | 73 | 92 |
| 5·5 | 1·70475 | 70656 | 70838 | 71019 | 71199 | 71380 | 71560 | 71740 | 71919 | 72098 | 18 | 36 | 54 | 72 | 90 |
| 5·6 | 1·72277 | 72455 | 72633 | 72811 | 72988 | 73166 | 73342 | 73519 | 73695 | 73871 | 18 | 35 | 53 | 71 | 89 |
| 5·7 | 1·74047 | 74222 | 74397 | 74572 | 74746 | 74920 | 75094 | 75267 | 75440 | 75613 | 17 | 35 | 52 | 70 | 87 |
| 5·8 | 1·75786 | 75958 | 76130 | 76302 | 76473 | 76644 | 76815 | 76985 | 77156 | 77326 | 17 | 34 | 51 | 68 | 86 |
| 5·9 | 1·77495 | 77665 | 77834 | 78002 | 78171 | 78339 | 78507 | 78675 | 78842 | 79009 | 17 | 34 | 50 | 67 | 84 |
| 6·0 | 1·79176 | 79342 | 79509 | 79675 | 79840 | 80006 | 80171 | 80336 | 80500 | 80665 | 17 | 33 | 50 | 66 | 83 |
| 6·1 | 1·80829 | 80993 | 81156 | 81319 | 81482 | 81645 | 81808 | 81970 | 82132 | 82294 | 16 | 33 | 49 | 65 | 81 |
| 6·2 | 1·82455 | 82616 | 82777 | 82938 | 83098 | 83258 | 83418 | 83578 | 83737 | 83896 | 16 | 32 | 48 | 64 | 80 |
| 6·3 | 1·84055 | 84214 | 84372 | 84530 | 84688 | 84845 | 85003 | 85160 | 85317 | 85473 | 16 | 32 | 47 | 63 | 79 |
| 6·4 | 1·85630 | 85786 | 85942 | 86097 | 86253 | 86408 | 86563 | 86718 | 86872 | 87026 | 16 | 31 | 47 | 62 | 78 |
| 6·5 | 1·87180 | 87334 | 87487 | 87641 | 87794 | 87947 | 88099 | 88251 | 88403 | 88555 | 15 | 31 | 46 | 61 | 76 |
| 6·6 | 1·88707 | 88858 | 89010 | 89160 | 89311 | 89462 | 89612 | 89762 | 89912 | 90061 | 15 | 30 | 45 | 60 | 75 |
| 6·7 | 1·90211 | 90360 | 90509 | 90658 | 90806 | 90954 | 91102 | 91250 | 91398 | 91545 | 15 | 30 | 44 | 59 | 74 |
| 6·8 | 1·91692 | 91839 | 91986 | 92132 | 92279 | 92425 | 92571 | 92716 | 92862 | 93007 | 15 | 29 | 44 | 58 | 73 |
| 6·9 | 1·93152 | 93297 | 93442 | 93586 | 93730 | 93874 | 94018 | 94162 | 94305 | 94448 | 14 | 29 | 43 | 58 | 72 |
| 7·0 | 1·94591 | 94734 | 94876 | 95019 | 95161 | 95303 | 95445 | 95586 | 95727 | 95869 | 14 | 28 | 43 | 57 | 71 |
| 7·1 | 1·96009 | 96150 | 96291 | 96431 | 96571 | 96711 | 96851 | 96991 | 97130 | 97269 | 14 | 28 | 42 | 56 | 70 |
| 7·2 | 1·97408 | 97547 | 97685 | 97824 | 97962 | 98100 | 98238 | 98376 | 98513 | 98650 | 14 | 28 | 41 | 55 | 69 |
| 7·3 | 1·98787 | 98924 | 99061 | 99198 | 99334 | 99470 | 99606 | 99742 | 99877 | **00013** | 14 | 27 | 41 | 54 | 68 |
| 7·4 | 2·00148 | 00283 | 00418 | 00553 | 00687 | 00821 | 00956 | 01089 | 01223 | 01357 | 13 | 27 | 40 | 54 | 67 |
| 7·5 | 2·01490 | 01624 | 01757 | 01890 | 02022 | 02155 | 02287 | 02419 | 02551 | 02683 | 13 | 27 | 40 | 53 | 66 |
| 7·6 | 2·02815 | 02946 | 03078 | 03209 | 03340 | 03471 | 03601 | 03732 | 03862 | 03992 | 13 | 26 | 39 | 52 | 65 |
| 7·7 | 2·04122 | 04252 | 04381 | 04511 | 04640 | 04769 | 04898 | 05027 | 05156 | 05284 | 13 | 26 | 39 | 52 | 65 |
| 7·8 | 2·05412 | 05540 | 05668 | 05796 | 05924 | 06051 | 06179 | 06306 | 06433 | 06560 | 13 | 25 | 38 | 51 | 64 |
| 7·9 | 2·06686 | 06813 | 06939 | 07065 | 07191 | 07317 | 07443 | 07568 | 07694 | 07819 | 13 | 25 | 38 | 50 | 63 |
| 8·0 | 2·07944 | 08069 | 08194 | 08318 | 08443 | 08567 | 08691 | 08815 | 08939 | 09063 | 12 | 25 | 37 | 50 | 62 |
| 8·1 | 2·09186 | 09310 | 09433 | 09556 | 09679 | 09802 | 09924 | 10047 | 10169 | 10291 | 12 | 25 | 37 | 49 | 61 |
| 8·2 | 2·10413 | 10535 | 10657 | 10779 | 10900 | 11021 | 11142 | 11263 | 11384 | 11505 | 12 | 24 | 36 | 49 | 61 |
| 8·3 | 2·11626 | 11746 | 11866 | 11986 | 12106 | 12226 | 12346 | 12465 | 12585 | 12704 | 12 | 24 | 36 | 48 | 60 |
| 8·4 | 2·12823 | 12942 | 13061 | 13180 | 13298 | 13417 | 13535 | 13653 | 13771 | 13889 | 12 | 24 | 36 | 47 | 59 |
| 8·5 | 2·14007 | 14124 | 14242 | 14359 | 14476 | 14593 | 14710 | 14827 | 14943 | 15060 | 12 | 23 | 35 | 47 | 58 |
| 8·6 | 2·15176 | 15292 | 15409 | 15524 | 15640 | 15756 | 15871 | 15987 | 16102 | 16217 | 12 | 23 | 35 | 46 | 58 |
| 8·7 | 2·16332 | 16447 | 16562 | 16677 | 16791 | 16905 | 17020 | 17134 | 17248 | 17361 | 11 | 23 | 34 | 46 | 57 |
| 8·8 | 2·17475 | 17589 | 17702 | 17816 | 17929 | 18042 | 18155 | 18267 | 18380 | 18493 | 11 | 23 | 34 | 45 | 56 |
| 8·9 | 2·18605 | 18717 | 18830 | 18942 | 19054 | 19165 | 19277 | 19389 | 19500 | 19611 | 11 | 22 | 34 | 45 | 56 |
| 9·0 | 2·19722 | 19834 | 19944 | 20055 | 20166 | 20276 | 20387 | 20497 | 20607 | 20717 | 11 | 22 | 33 | 44 | 55 |
| 9·1 | 2·20827 | 20937 | 21047 | 21157 | 21266 | 21375 | 21485 | 21594 | 21703 | 21812 | 11 | 22 | 33 | 44 | 55 |
| 9·2 | 2·21920 | 22029 | 22138 | 22246 | 22354 | 22462 | 22570 | 22678 | 22786 | 22894 | 11 | 22 | 32 | 43 | 54 |
| 9·3 | 2·23001 | 23109 | 23216 | 23324 | 23431 | 23538 | 23645 | 23751 | 23858 | 23965 | 11 | 21 | 32 | 43 | 54 |
| 9·4 | 2·24071 | 24177 | 24284 | 24390 | 24496 | 24601 | 24707 | 24813 | 24918 | 25024 | 11 | 21 | 32 | 42 | 53 |
| 9·5 | 2·25129 | 25234 | 25339 | 25444 | 25549 | 25654 | 25759 | 25863 | 25968 | 26072 | 10 | 21 | 31 | 42 | 52 |
| 9·6 | 2·26176 | 26280 | 26384 | 26488 | 26592 | 26696 | 26799 | 26903 | 27006 | 27109 | 10 | 21 | 31 | 41 | 52 |
| 9·7 | 2·27213 | 27316 | 27419 | 27521 | 27624 | 27727 | 27829 | 27932 | 28034 | 28136 | 10 | 21 | 31 | 41 | 51 |
| 9·8 | 2·28238 | 28340 | 28442 | 28544 | 28646 | 28747 | 28849 | 28950 | 29051 | 29152 | 10 | 20 | 30 | 41 | 51 |
| 9·9 | 2·29253 | 29354 | 29455 | 29556 | 29657 | 29757 | 29858 | 29958 | 30058 | 30158 | 10 | 20 | 30 | 40 | 50 |

$\log_e 10{,}000 = 9{\cdot}21034$, $\log_e 100{,}000 = 11{\cdot}51293$, $\log_e 1{,}000{,}000 = 13{\cdot}81551$

## TABLE XXVI. NATURAL LOGARITHMS—*continued*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 2·30259 | 31254 | 32239 | 33214 | 34181 | — | — | — | — | — | 98 | 196 | 294 | 392 | 490 |
| | — | — | — | — | — | 35138 | 36085 | 37024 | 37955 | 38876 | 93 | 187 | 280 | 374 | 467 |
| 11 | 2·39790 | 40695 | 41591 | 42480 | 43361 | — | — | — | — | — | 89 | 179 | 268 | 357 | 446 |
| | — | — | — | — | — | 44235 | 45101 | 45959 | 46810 | 47654 | 85 | 171 | 256 | 342 | 427 |
| 12 | 2·48491 | 49321 | 50144 | 50960 | 51770 | — | — | — | — | — | 82 | 164 | 246 | 328 | 410 |
| | — | — | — | — | — | 52573 | 53370 | 54160 | 54945 | 55723 | 79 | 158 | 236 | 315 | 394 |
| 13 | 2·56495 | 57261 | 58022 | 58776 | 59525 | — | — | — | — | — | 76 | 152 | 227 | 303 | 379 |
| | — | — | — | — | — | 60269 | 61007 | 61740 | 62467 | 63189 | 73 | 146 | 219 | 292 | 365 |
| 14 | 2·63906 | 64617 | 65324 | 66026 | 66723 | — | — | — | — | — | 70 | 141 | 211 | 282 | 352 |
| | — | — | — | — | — | 67415 | 68102 | 68785 | 69463 | 70136 | 68 | 136 | 204 | 272 | 340 |
| 15 | 2·70805 | 71469 | 72130 | 72785 | 73437 | — | — | — | — | — | 66 | 132 | 197 | 263 | 329 |
| | — | — | — | — | — | 74084 | 74727 | 75366 | 76001 | 76632 | 64 | 127 | 191 | 255 | 318 |
| 16 | 2·77259 | 77882 | 78501 | 79117 | 79728 | — | — | — | — | — | 62 | 124 | 185 | 247 | 309 |
| | — | — | — | — | — | 80336 | 80940 | 81541 | 82138 | 82731 | 60 | 120 | 180 | 240 | 299 |
| 17 | 2·83321 | 83908 | 84491 | 85071 | 85647 | — | — | — | — | — | 58 | 116 | 174 | 233 | 291 |
| | — | — | — | — | — | 86220 | 86790 | 87356 | 87920 | 88480 | 56 | 113 | 169 | 226 | 282 |
| 18 | 2·89037 | 89591 | 90142 | 90690 | 91235 | — | — | — | — | — | 55 | 110 | 165 | 220 | 275 |
| | — | — | — | — | — | 91777 | 92316 | 92852 | 93386 | 93916 | 53 | 107 | 160 | 214 | 267 |
| 19 | 2·94444 | 94969 | 95491 | 96011 | 96527 | — | — | — | — | — | 52 | 104 | 156 | 208 | 260 |
| | — | — | — | — | — | 97041 | 97553 | 98062 | 98568 | 99072 | 51 | 102 | 152 | 203 | 254 |
| 20 | 2·99573 | **00072** | **00568** | **01062** | **01553** | **02042** | **02529** | **03013** | **03495** | **03975** | 49 | 98 | 147 | 196 | 244 |
| 21 | 3·04452 | 04927 | 05400 | 05871 | 06339 | 06805 | 07269 | 07731 | 08191 | 08649 | 47 | 93 | 140 | 186 | 233 |
| 22 | 3·09104 | 09558 | 10009 | 10459 | 10906 | 11352 | 11795 | 12237 | 12676 | 13114 | 45 | 89 | 134 | 178 | 223 |
| 23 | 3·13549 | 13983 | 14415 | 14845 | 15274 | 15700 | 16125 | 16548 | 16969 | 17388 | 43 | 85 | 128 | 171 | 213 |
| 24 | 3·17805 | 18221 | 18635 | 19048 | 19458 | 19867 | 20275 | 20680 | 21084 | 21487 | 41 | 82 | 123 | 164 | 204 |
| 25 | 3·21888 | 22287 | 22684 | 23080 | 23475 | 23868 | 24259 | 24649 | 25037 | 25424 | 39 | 79 | 118 | 157 | 196 |
| 26 | 3·25810 | 26194 | 26576 | 26957 | 27336 | 27714 | 28091 | 28466 | 28840 | 29213 | 38 | 76 | 113 | 151 | 189 |
| 27 | 3·29584 | 29953 | 30322 | 30689 | 31054 | 31419 | 31782 | 32143 | 32504 | 32863 | 36 | 73 | 109 | 146 | 182 |
| 28 | 3·33220 | 33577 | 33932 | 34286 | 34639 | 34990 | 35341 | 35690 | 36038 | 36384 | 35 | 70 | 105 | 141 | 176 |
| 29 | 3·36730 | 37074 | 37417 | 37759 | 38099 | 38439 | 38777 | 39115 | 39451 | 39786 | 34 | 68 | 102 | 136 | 170 |
| 30 | 3·40120 | 40453 | 40784 | 41115 | 41444 | 41773 | 42100 | 42426 | 42751 | 43076 | 33 | 66 | 99 | 131 | 164 |
| 31 | 3·43399 | 43721 | 44042 | 44362 | 44681 | 44999 | 45316 | 45632 | 45947 | 46261 | 32 | 64 | 95 | 127 | 159 |
| 32 | 3·46574 | 46886 | 47197 | 47507 | 47816 | 48124 | 48431 | 48738 | 49043 | 49347 | 31 | 62 | 92 | 123 | 154 |
| 33 | 3·49651 | 49953 | 50255 | 50556 | 50856 | 51155 | 51453 | 51750 | 52046 | 52342 | 30 | 60 | 90 | 120 | 150 |
| 34 | 3·52636 | 52930 | 53223 | 53515 | 53806 | 54096 | 54385 | 54674 | 54962 | 55249 | 29 | 58 | 87 | 116 | 145 |
| 35 | 3·55535 | 55820 | 56105 | 56388 | 56671 | 56953 | 57235 | 57515 | 57795 | 58074 | 28 | 56 | 85 | 113 | 141 |
| 36 | 3·58352 | 58629 | 58906 | 59182 | 59457 | 59731 | 60005 | 60278 | 60550 | 60821 | 27 | 55 | 82 | 110 | 137 |
| 37 | 3·61092 | 61362 | 61631 | 61899 | 62167 | 62434 | 62700 | 62966 | 63231 | 63495 | 27 | 53 | 80 | 107 | 134 |
| 38 | 3·63759 | 64021 | 64284 | 64545 | 64806 | 65066 | 65325 | 65584 | 65842 | 66099 | 26 | 52 | 78 | 104 | 130 |
| 39 | 3·66356 | 66612 | 66868 | 67122 | 67377 | 67630 | 67883 | 68135 | 68387 | 68638 | 25 | 51 | 76 | 101 | 127 |
| 40 | 3·68888 | 69138 | 69387 | 69635 | 69883 | 70130 | 70377 | 70623 | 70868 | 71113 | 25 | 49 | 74 | 99 | 124 |
| 41 | 3·71357 | 71601 | 71844 | 72086 | 72328 | 72569 | 72810 | 73050 | 73290 | 73529 | 24 | 48 | 72 | 96 | 121 |
| 42 | 3·73767 | 74005 | 74242 | 74479 | 74715 | 74950 | 75185 | 75420 | 75654 | 75887 | 24 | 47 | 71 | 94 | 118 |
| 43 | 3·76120 | 76352 | 76584 | 76815 | 77046 | 77276 | 77506 | 77735 | 77963 | 78191 | 23 | 46 | 69 | 92 | 115 |
| 44 | 3·78419 | 78646 | 78872 | 79098 | 79324 | 79549 | 79773 | 79997 | 80221 | 80444 | 22 | 45 | 68 | 90 | 112 |
| 45 | 3·80666 | 80888 | 81110 | 81331 | 81551 | 81771 | 81991 | 82210 | 82428 | 82647 | 22 | 44 | 66 | 88 | 110 |
| 46 | 3·82864 | 83081 | 83298 | 83514 | 83730 | 83945 | 84160 | 84374 | 84588 | 84802 | 22 | 43 | 65 | 86 | 108 |
| 47 | 3·85015 | 85227 | 85439 | 85651 | 85862 | 86073 | 86283 | 86493 | 86703 | 86912 | 21 | 42 | 63 | 84 | 105 |
| 48 | 3·87120 | 87328 | 87536 | 87743 | 87950 | 88156 | 88362 | 88568 | 88773 | 88978 | 21 | 41 | 62 | 83 | 103 |
| 49 | 3·89182 | 89386 | 89589 | 89792 | 89995 | 90197 | 90399 | 90600 | 90801 | 91002 | 20 | 40 | 61 | 81 | 101 |

$\log_e 10 = 2{\cdot}30259$, $\log_e 100 = 4{\cdot}60517$, $\log_e 1000 = 6{\cdot}90776$

# TABLE XXVI. NATURAL LOGARITHMS—*continued*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 3·91202 | 91402 | 91601 | 91801 | 91999 | 92197 | 92395 | 92593 | 92790 | 92986 | 20 | 40 | 59 | 79 | 99 |
| 51 | 3·93183 | 93378 | 93574 | 93769 | 93964 | 94158 | 94352 | 94546 | 94739 | 94932 | 19 | 39 | 58 | 78 | 97 |
| 52 | 3·95124 | 95316 | 95508 | 95700 | 95891 | 96081 | 96272 | 96462 | 96651 | 96840 | 19 | 38 | 57 | 76 | 95 |
| 53 | 3·97029 | 97218 | 97406 | 97594 | 97781 | 97968 | 98155 | 98341 | 98527 | 98713 | 19 | 37 | 56 | 75 | 94 |
| 54 | 3·98898 | 99083 | 99268 | 99452 | 99636 | 99820 | **00003** | **00186** | **00369** | **00551** | 18 | 37 | 55 | 73 | 92 |
| 55 | 4·00733 | 00915 | 01096 | 01277 | 01458 | 01638 | 01818 | 01998 | 02177 | 02356 | 18 | 36 | 54 | 72 | 90 |
| 56 | 4·02535 | 02714 | 02892 | 03069 | 03247 | 03424 | 03601 | 03777 | 03954 | 04130 | 18 | 35 | 53 | 71 | 89 |
| 57 | 4·04305 | 04480 | 04655 | 04830 | 05004 | 05178 | 05352 | 05526 | 05699 | 05872 | 17 | 35 | 52 | 70 | 87 |
| 58 | 4·06044 | 06217 | 06389 | 06560 | 06732 | 06903 | 07073 | 07244 | 07414 | 07584 | 17 | 34 | 51 | 68 | 86 |
| 59 | 4·07754 | 07923 | 08092 | 08261 | 08429 | 08598 | 08766 | 08933 | 09101 | 09268 | 17 | 34 | 50 | 67 | 84 |
| 60 | 4·09434 | 09601 | 09767 | 09933 | 10099 | 10264 | 10429 | 10594 | 10759 | 10923 | 17 | 33 | 50 | 66 | 83 |
| 61 | 4·11087 | 11251 | 11415 | 11578 | 11741 | 11904 | 12066 | 12228 | 12390 | 12552 | 16 | 33 | 49 | 65 | 81 |
| 62 | 4·12713 | 12875 | 13036 | 13196 | 13357 | 13517 | 13677 | 13836 | 13995 | 14155 | 16 | 32 | 48 | 64 | 80 |
| 63 | 4·14313 | 14472 | 14630 | 14789 | 14946 | 15104 | 15261 | 15418 | 15575 | 15732 | 16 | 32 | 47 | 63 | 79 |
| 64 | 4·15888 | 16044 | 16200 | 16356 | 16511 | 16667 | 16821 | 16976 | 17131 | 17285 | 16 | 31 | 47 | 62 | 78 |
| 65 | 4·17439 | 17592 | 17746 | 17899 | 18052 | 18205 | 18358 | 18510 | 18662 | 18814 | 15 | 31 | 46 | 61 | 76 |
| 66 | 4·18965 | 19117 | 19268 | 19419 | 19570 | 19720 | 19870 | 20020 | 20170 | 20320 | 15 | 30 | 45 | 60 | 75 |
| 67 | 4·20469 | 20618 | 20767 | 20916 | 21065 | 21213 | 21361 | 21509 | 21656 | 21804 | 15 | 30 | 44 | 59 | 74 |
| 68 | 4·21951 | 22098 | 22244 | 22391 | 22537 | 22683 | 22829 | 22975 | 23120 | 23266 | 15 | 29 | 44 | 58 | 73 |
| 69 | 4·23411 | 23555 | 23700 | 23844 | 23989 | 24133 | 24276 | 24420 | 24563 | 24707 | 14 | 29 | 43 | 58 | 72 |
| 70 | 4·24850 | 24992 | 25135 | 25277 | 25419 | 25561 | 25703 | 25845 | 25986 | 26127 | 14 | 28 | 43 | 57 | 71 |
| 71 | 4·26268 | 26409 | 26549 | 26690 | 26830 | 26970 | 27110 | 27249 | 27388 | 27528 | 14 | 28 | 42 | 56 | 70 |
| 72 | 4·27667 | 27805 | 27944 | 28082 | 28221 | 28359 | 28496 | 28634 | 28772 | 28909 | 14 | 28 | 41 | 55 | 69 |
| 73 | 4·29046 | 29183 | 29320 | 29456 | 29592 | 29729 | 29865 | 30000 | 30136 | 30271 | 14 | 27 | 41 | 54 | 68 |
| 74 | 4·30407 | 30542 | 30676 | 30811 | 30946 | 31080 | 31214 | 31348 | 31482 | 31615 | 13 | 27 | 40 | 54 | 67 |
| 75 | 4·31749 | 31882 | 32015 | 32148 | 32281 | 32413 | 32546 | 32678 | 32810 | 32942 | 13 | 27 | 40 | 53 | 66 |
| 76 | 4·33073 | 33205 | 33336 | 33467 | 33598 | 33729 | 33860 | 33990 | 34120 | 34251 | 13 | 26 | 39 | 52 | 65 |
| 77 | 4·34381 | 34510 | 34640 | 34769 | 34899 | 35028 | 35157 | 35286 | 35414 | 35543 | 13 | 26 | 39 | 52 | 65 |
| 78 | 4·35671 | 35799 | 35927 | 36055 | 36182 | 36310 | 36437 | 36564 | 36691 | 36818 | 13 | 25 | 38 | 51 | 64 |
| 79 | 4·36945 | 37071 | 37198 | 37324 | 37450 | 37576 | 37701 | 37827 | 37952 | 38078 | 13 | 25 | 38 | 50 | 63 |
| 80 | 4·38203 | 38328 | 38452 | 38577 | 38701 | 38826 | 38950 | 39074 | 39198 | 39321 | 12 | 25 | 37 | 50 | 62 |
| 81 | 4·39445 | 39568 | 39692 | 39815 | 39938 | 40060 | 40183 | 40305 | 40428 | 40550 | 12 | 25 | 37 | 49 | 61 |
| 82 | 4·40672 | 40794 | 40916 | 41037 | 41159 | 41280 | 41401 | 41522 | 41643 | 41763 | 12 | 24 | 36 | 49 | 61 |
| 83 | 4·41884 | 42004 | 42125 | 42245 | 42365 | 42485 | 42604 | 42724 | 42843 | 42963 | 12 | 24 | 36 | 48 | 60 |
| 84 | 4·43082 | 43201 | 43319 | 43438 | 43557 | 43675 | 43793 | 43912 | 44030 | 44147 | 12 | 24 | 36 | 47 | 59 |
| 85 | 4·44265 | 44383 | 44500 | 44617 | 44735 | 44852 | 44969 | 45085 | 45202 | 45318 | 12 | 23 | 35 | 47 | 58 |
| 86 | 4·45435 | 45551 | 45667 | 45783 | 45899 | 46014 | 46130 | 46245 | 46361 | 46476 | 12 | 23 | 35 | 46 | 58 |
| 87 | 4·46591 | 46706 | 46820 | 46935 | 47050 | 47164 | 47278 | 47392 | 47506 | 47620 | 11 | 23 | 34 | 46 | 57 |
| 88 | 4·47734 | 47847 | 47961 | 48074 | 48187 | 48300 | 48413 | 48526 | 48639 | 48751 | 11 | 23 | 34 | 45 | 56 |
| 89 | 4·48864 | 48976 | 49088 | 49200 | 49312 | 49424 | 49536 | 49647 | 49758 | 49870 | 11 | 22 | 34 | 45 | 56 |
| 90 | 4·49981 | 50092 | 50203 | 50314 | 50424 | 50535 | 50645 | 50756 | 50866 | 50976 | 11 | 22 | 33 | 44 | 55 |
| 91 | 4·51086 | 51196 | 51305 | 51415 | 51525 | 51634 | 51743 | 51852 | 51961 | 52070 | 11 | 22 | 33 | 44 | 55 |
| 92 | 4·52179 | 52287 | 52396 | 52504 | 52613 | 52721 | 52829 | 52937 | 53045 | 53152 | 11 | 22 | 32 | 43 | 54 |
| 93 | 4·53260 | 53367 | 53475 | 53582 | 53689 | 53796 | 53903 | 54010 | 54116 | 54223 | 11 | 21 | 32 | 43 | 54 |
| 94 | 4·54329 | 54436 | 54542 | 54648 | 54754 | 54860 | 54966 | 55071 | 55177 | 55282 | 11 | 21 | 32 | 42 | 53 |
| 95 | 4·55388 | 55493 | 55598 | 55703 | 55808 | 55913 | 56017 | 56122 | 56226 | 56331 | 10 | 21 | 31 | 42 | 52 |
| 96 | 4·56435 | 56539 | 56643 | 56747 | 56851 | 56954 | 57058 | 57161 | 57265 | 57368 | 10 | 21 | 31 | 41 | 52 |
| 97 | 4·57471 | 57574 | 57677 | 57780 | 57883 | 57985 | 58088 | 58190 | 58292 | 58395 | 10 | 21 | 31 | 41 | 51 |
| 98 | 4·58497 | 58599 | 58701 | 58802 | 58904 | 59006 | 59107 | 59209 | 59310 | 59411 | 10 | 20 | 30 | 41 | 51 |
| 99 | 4·59512 | 59613 | 59714 | 59815 | 59915 | 60016 | 60116 | 60217 | 60317 | 60417 | 10 | 20 | 30 | 40 | 50 |

$\log_e 10{,}000 = 9{\cdot}21034$, $\log_e 100{,}000 = 11{\cdot}51293$. $\log_e 1{,}000{,}000 = 13{\cdot}81551$ .

# TABLE XXVII. SQUARES

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 |
| 1 | 100 | 121 | 144 | 169 | 196 | 225 | 256 | 289 | 324 | 361 |
| 2 | 400 | 441 | 484 | 529 | 576 | 625 | 676 | 729 | 784 | 841 |
| 3 | 900 | 961 | 1024 | 1089 | 1156 | 1225 | 1296 | 1369 | 1444 | 1521 |
| 4 | 1600 | 1681 | 1764 | 1849 | 1936 | 2025 | 2116 | 2209 | 2304 | 2401 |
| 5 | 2500 | 2601 | 2704 | 2809 | 2916 | 3025 | 3136 | 3249 | 3364 | 3481 |
| 6 | 3600 | 3721 | 3844 | 3969 | 4096 | 4225 | 4356 | 4489 | 4624 | 4761 |
| 7 | 4900 | 5041 | 5184 | 5329 | 5476 | 5625 | 5776 | 5929 | 6084 | 6241 |
| 8 | 6400 | 6561 | 6724 | 6889 | 7056 | 7225 | 7396 | 7569 | 7744 | 7921 |
| 9 | 8100 | 8281 | 8464 | 8649 | 8836 | 9025 | 9216 | 9409 | 9604 | 9801 |
| 10 | 10000 | 10201 | 10404 | 10609 | 10816 | 11025 | 11236 | 11449 | 11664 | 11881 |
| 11 | 12100 | 12321 | 12544 | 12769 | 12996 | 13225 | 13456 | 13689 | 13924 | 14161 |
| 12 | 14400 | 14641 | 14884 | 15129 | 15376 | 15625 | 15876 | 16129 | 16384 | 16641 |
| 13 | 16900 | 17161 | 17424 | 17689 | 17956 | 18225 | 18496 | 18769 | 19044 | 19321 |
| 14 | 19600 | 19881 | 20164 | 20449 | 20736 | 21025 | 21316 | 21609 | 21904 | 22201 |
| 15 | 22500 | 22801 | 23104 | 23409 | 23716 | 24025 | 24336 | 24649 | 24964 | 25281 |
| 16 | 25600 | 25921 | 26244 | 26569 | 26896 | 27225 | 27556 | 27889 | 28224 | 28561 |
| 17 | 28900 | 29241 | 29584 | 29929 | 30276 | 30625 | 30976 | 31329 | 31684 | 32041 |
| 18 | 32400 | 32761 | 33124 | 33489 | 33856 | 34225 | 34596 | 34969 | 35344 | 35721 |
| 19 | 36100 | 36481 | 36864 | 37249 | 37636 | 38025 | 38416 | 38809 | 39204 | 39601 |
| 20 | 40000 | 40401 | 40804 | 41209 | 41616 | 42025 | 42436 | 42849 | 43264 | 43681 |
| 21 | 44100 | 44521 | 44944 | 45369 | 45796 | 46225 | 46656 | 47089 | 47524 | 47961 |
| 22 | 48400 | 48841 | 49284 | 49729 | 50176 | 50625 | 51076 | 51529 | 51984 | 52441 |
| 23 | 52900 | 53361 | 53824 | 54289 | 54756 | 55225 | 55696 | 56169 | 56644 | 57121 |
| 24 | 57600 | 58081 | 58564 | 59049 | 59536 | 60025 | 60516 | 61009 | 61504 | 62001 |
| 25 | 62500 | 63001 | 63504 | 64009 | 64516 | 65025 | 65536 | 66049 | 66564 | 67081 |
| 26 | 67600 | 68121 | 68644 | 69169 | 69696 | 70225 | 70756 | 71289 | 71824 | 72361 |
| 27 | 72900 | 73441 | 73984 | 74529 | 75076 | 75625 | 76176 | 76729 | 77284 | 77841 |
| 28 | 78400 | 78961 | 79524 | 80089 | 80656 | 81225 | 81796 | 82369 | 82944 | 83521 |
| 29 | 84100 | 84681 | 85264 | 85849 | 86436 | 87025 | 87616 | 88209 | 88804 | 89401 |
| 30 | 90000 | 90601 | 91204 | 91809 | 92416 | 93025 | 93636 | 94249 | 94864 | 95481 |
| 31 | 96100 | 96721 | 97344 | 97969 | 98596 | 99225 | 99856 | 100489 | 101124 | 101761 |
| 32 | 102400 | 103041 | 103684 | 104329 | 104976 | 105625 | 106276 | 106929 | 107584 | 108241 |
| 33 | 108900 | 109561 | 110224 | 110889 | 111556 | 112225 | 112896 | 113569 | 114244 | 114921 |
| 34 | 115600 | 116281 | 116964 | 117649 | 118336 | 119025 | 119716 | 120409 | 121104 | 121801 |
| 35 | 122500 | 123201 | 123904 | 124609 | 125316 | 126025 | 126736 | 127449 | 128164 | 128881 |
| 36 | 129600 | 130321 | 131044 | 131769 | 132496 | 133225 | 133956 | 134689 | 135424 | 136161 |
| 37 | 136900 | 137641 | 138384 | 139129 | 139876 | 140625 | 141376 | 142129 | 142884 | 143641 |
| 38 | 144400 | 145161 | 145924 | 146689 | 147456 | 148225 | 148996 | 149769 | 150544 | 151321 |
| 39 | 152100 | 152881 | 153664 | 154449 | 155236 | 156025 | 156816 | 157609 | 158404 | 159201 |
| 40 | 160000 | 160801 | 161604 | 162409 | 163216 | 164025 | 164836 | 165649 | 166464 | 167281 |
| 41 | 168100 | 168921 | 169744 | 170569 | 171396 | 172225 | 173056 | 173889 | 174724 | 175561 |
| 42 | 176400 | 177241 | 178084 | 178929 | 179776 | 180625 | 181476 | 182329 | 183184 | 184041 |
| 43 | 184900 | 185761 | 186624 | 187489 | 188356 | 189225 | 190096 | 190969 | 191844 | 192721 |
| 44 | 193600 | 194481 | 195364 | 196249 | 197136 | 198025 | 198916 | 199809 | 200704 | 201601 |
| 45 | 202500 | 203401 | 204304 | 205209 | 206116 | 207025 | 207936 | 208849 | 209764 | 210681 |
| 46 | 211600 | 212521 | 213444 | 214369 | 215296 | 216225 | 217156 | 218089 | 219024 | 219961 |
| 47 | 220900 | 221841 | 222784 | 223729 | 224676 | 225625 | 226576 | 227529 | 228484 | 229441 |
| 48 | 230400 | 231361 | 232324 | 233289 | 234256 | 235225 | 236196 | 237169 | 238144 | 239121 |
| 49 | 240100 | 241081 | 242064 | 243049 | 244036 | 245025 | 246016 | 247009 | 248004 | 249001 |

Exact squares of 4 figure numbers can be quickly calculated from the identity $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 250000 | 251001 | 252004 | 253009 | 254016 | 255025 | 256036 | 257049 | 258064 | 259081 |
| 51 | 260100 | 261121 | 262144 | 263169 | 264196 | 265225 | 266256 | 267289 | 268324 | 269361 |
| 52 | 270400 | 271441 | 272484 | 273529 | 274576 | 275625 | 276676 | 277729 | 278784 | 279841 |
| 53 | 280900 | 281961 | 283024 | 284089 | 285156 | 286225 | 287296 | 288369 | 289444 | 290521 |
| 54 | 291600 | 292681 | 293764 | 294849 | 295936 | 297025 | 298116 | 299209 | 300304 | 301401 |
| 55 | 302500 | 303601 | 304704 | 305809 | 306916 | 308025 | 309136 | 310249 | 311364 | 312481 |
| 56 | 313600 | 314721 | 315844 | 316969 | 318096 | 319225 | 320356 | 321489 | 322624 | 323761 |
| 57 | 324900 | 326041 | 327184 | 328329 | 329476 | 330625 | 331776 | 332929 | 334084 | 335241 |
| 58 | 336400 | 337561 | 338724 | 339889 | 341056 | 342225 | 343396 | 344569 | 345744 | 346921 |
| 59 | 348100 | 349281 | 350464 | 351649 | 352836 | 354025 | 355216 | 356409 | 357604 | 358801 |
| 60 | 360000 | 361201 | 362404 | 363609 | 364816 | 366025 | 367236 | 368449 | 369664 | 370881 |
| 61 | 372100 | 373321 | 374544 | 375769 | 376996 | 378225 | 379456 | 380689 | 381924 | 383161 |
| 62 | 384400 | 385641 | 386884 | 388129 | 389376 | 390625 | 391876 | 393129 | 394384 | 395641 |
| 63 | 396900 | 398161 | 399424 | 400689 | 401956 | 403225 | 404496 | 405769 | 407044 | 408321 |
| 64 | 409600 | 410881 | 412164 | 413449 | 414736 | 416025 | 417316 | 418609 | 419904 | 421201 |
| 65 | 422500 | 423801 | 425104 | 426409 | 427716 | 429025 | 430336 | 431649 | 432964 | 434281 |
| 66 | 435600 | 436921 | 438244 | 439569 | 440896 | 442225 | 443556 | 444889 | 446224 | 447561 |
| 67 | 448900 | 450241 | 451584 | 452929 | 454276 | 455625 | 456976 | 458329 | 459684 | 461041 |
| 68 | 462400 | 463761 | 465124 | 466489 | 467856 | 469225 | 470596 | 471969 | 473344 | 474721 |
| 69 | 476100 | 477481 | 478864 | 480249 | 481636 | 483025 | 484416 | 485809 | 487204 | 488601 |
| 70 | 490000 | 491401 | 492804 | 494209 | 495616 | 497025 | 498436 | 499849 | 501264 | 502681 |
| 71 | 504100 | 505521 | 506944 | 508369 | 509796 | 511225 | 512656 | 514089 | 515524 | 516961 |
| 72 | 518400 | 519841 | 521284 | 522729 | 524176 | 525625 | 527076 | 528529 | 529984 | 531441 |
| 73 | 532900 | 534361 | 535824 | 537289 | 538756 | 540225 | 541696 | 543169 | 544644 | 546121 |
| 74 | 547600 | 549081 | 550564 | 552049 | 553536 | 555025 | 556516 | 558009 | 559504 | 561001 |
| 75 | 562500 | 564001 | 565504 | 567009 | 568516 | 570025 | 571536 | 573049 | 574564 | 576081 |
| 76 | 577600 | 579121 | 580644 | 582169 | 583696 | 585225 | 586756 | 588289 | 589824 | 591361 |
| 77 | 592900 | 594441 | 595984 | 597529 | 599076 | 600625 | 602176 | 603729 | 605284 | 606841 |
| 78 | 608400 | 609961 | 611524 | 613089 | 614656 | 616225 | 617796 | 619369 | 620944 | 622521 |
| 79 | 624100 | 625681 | 627264 | 628849 | 630436 | 632025 | 633616 | 635209 | 636804 | 638401 |
| 80 | 640000 | 641601 | 643204 | 644809 | 646416 | 648025 | 649636 | 651249 | 652864 | 654481 |
| 81 | 656100 | 657721 | 659344 | 660969 | 662596 | 664225 | 665856 | 667489 | 669124 | 670761 |
| 82 | 672400 | 674041 | 675684 | 677329 | 678976 | 680625 | 682276 | 683929 | 685584 | 687241 |
| 83 | 688900 | 690561 | 692224 | 693889 | 695556 | 697225 | 698896 | 700569 | 702244 | 703921 |
| 84 | 705600 | 707281 | 708964 | 710649 | 712336 | 714025 | 715716 | 717409 | 719104 | 720801 |
| 85 | 722500 | 724201 | 725904 | 727609 | 729316 | 731025 | 732736 | 734449 | 736164 | 737881 |
| 86 | 739600 | 741321 | 743044 | 744769 | 746496 | 748225 | 749956 | 751689 | 753424 | 755161 |
| 87 | 756900 | 758641 | 760384 | 762129 | 763876 | 765625 | 767376 | 769129 | 770884 | 772641 |
| 88 | 774400 | 776161 | 777924 | 779689 | 781456 | 783225 | 784996 | 786769 | 788544 | 790321 |
| 89 | 792100 | 793881 | 795664 | 797449 | 799236 | 801025 | 802816 | 804609 | 806404 | 808201 |
| 90 | 810000 | 811801 | 813604 | 815409 | 817216 | 819025 | 820836 | 822649 | 824464 | 826281 |
| 91 | 828100 | 829921 | 831744 | 833569 | 835396 | 837225 | 839056 | 840889 | 842724 | 844561 |
| 92 | 846400 | 848241 | 850084 | 851929 | 853776 | 855625 | 857476 | 859329 | 861184 | 863041 |
| 93 | 864900 | 866761 | 868624 | 870489 | 872356 | 874225 | 876096 | 877969 | 879844 | 881721 |
| 94 | 883600 | 885481 | 887364 | 889249 | 891136 | 893025 | 894916 | 896809 | 898704 | 900601 |
| 95 | 902500 | 904401 | 906304 | 908209 | 910116 | 912025 | 913936 | 915849 | 917764 | 919681 |
| 96 | 921600 | 923521 | 925444 | 927369 | 929296 | 931225 | 933156 | 935089 | 937024 | 938961 |
| 97 | 940900 | 942841 | 944784 | 946729 | 948676 | 950625 | 952576 | 954529 | 956484 | 958441 |
| 98 | 960400 | 962361 | 964324 | 966289 | 968256 | 970225 | 972196 | 974169 | 976144 | 978121 |
| 99 | 980100 | 982081 | 984064 | 986049 | 988036 | 990025 | 992016 | 994009 | 996004 | 998001 |

Thus $693{\cdot}3^2 = 480249 + 415{\cdot}8 + 0{\cdot}09 = 480664{\cdot}89$.

# TABLE XXVIII. SQUARE ROOTS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 10000 | 10050 | 10100 | 10149 | 10198 | 10247 | 10296 | 10344 | 10392 | 10440 | 5 | 10 | 15 | 20 | 24 |
| | 31623 | 31780 | 31937 | 32094 | 32249 | 32404 | 32558 | 32711 | 32863 | 33015 | 15 | 31 | 46 | 62 | 77 |
| 11 | 10488 | 10536 | 10583 | 10630 | 10677 | 10724 | 10770 | 10817 | 10863 | 10909 | 5 | 9 | 14 | 19 | 23 |
| | 33166 | 33317 | 33466 | 33615 | 33764 | 33912 | 34059 | 34205 | 34351 | 34496 | 15 | 30 | 44 | 59 | 74 |
| 12 | 10954 | 11000 | 11045 | 11091 | 11136 | 11180 | 11225 | 11269 | 11314 | 11358 | 4 | 9 | 13 | 18 | 22 |
| | 34641 | 34785 | 34928 | 35071 | 35214 | 35355 | 35496 | 35637 | 35777 | 35917 | 14 | 28 | 43 | 57 | 71 |
| 13 | 11402 | 11446 | 11489 | 11533 | 11576 | 11619 | 11662 | 11705 | 11747 | 11790 | 4 | 9 | 13 | 17 | 22 |
| | 36056 | 36194 | 36332 | 36469 | 36606 | 36742 | 36878 | 37014 | 37148 | 37283 | 14 | 27 | 41 | 55 | 68 |
| 14 | 11832 | 11874 | 11916 | 11958 | 12000 | 12042 | 12083 | 12124 | 12166 | 12207 | 4 | 8 | 13 | 17 | 21 |
| | 37417 | 37550 | 37683 | 37815 | 37947 | 38079 | 38210 | 38341 | 38471 | 38601 | 13 | 26 | 39 | 53 | 66 |
| 15 | 12247 | 12288 | 12329 | 12369 | 12410 | 12450 | 12490 | 12530 | 12570 | 12610 | 4 | 8 | 12 | 16 | 20 |
| | 38730 | 38859 | 38987 | 39115 | 39243 | 39370 | 39497 | 39623 | 39749 | 39875 | 13 | 25 | 38 | 51 | 64 |
| 16 | 12649 | 12689 | 12728 | 12767 | 12806 | 12845 | 12884 | 12923 | 12961 | 13000 | 4 | 8 | 12 | 16 | 20 |
| | 40000 | 40125 | 40249 | 40373 | 40497 | 40620 | 40743 | 40866 | 40988 | 41110 | 12 | 25 | 37 | 49 | 62 |
| 17 | 13038 | 13077 | 13115 | 13153 | 13191 | 13229 | 13266 | 13304 | 13342 | 13379 | 4 | 8 | 11 | 15 | 19 |
| | 41231 | 41352 | 41473 | 41593 | 41713 | 41833 | 41952 | 42071 | 42190 | 42308 | 12 | 24 | 36 | 48 | 60 |
| 18 | 13416 | 13454 | 13491 | 13528 | 13565 | 13601 | 13638 | 13675 | 13711 | 13748 | 4 | 7 | 11 | 15 | 18 |
| | 42426 | 42544 | 42661 | 42778 | 42895 | 43012 | 43128 | 43243 | 43359 | 43474 | 12 | 23 | 35 | 47 | 58 |
| 19 | 13784 | 13820 | 13856 | 13892 | 13928 | 13964 | 14000 | 14036 | 14071 | 14107 | 4 | 7 | 11 | 14 | 18 |
| | 43589 | 43704 | 43818 | 43932 | 44045 | 44159 | 44272 | 44385 | 44497 | 44609 | 11 | 23 | 34 | 45 | 57 |
| 20 | 14142 | 14177 | 14213 | 14248 | 14283 | 14318 | 14353 | 14387 | 14422 | 14457 | 4 | 7 | 10 | 14 | 18 |
| | 44721 | 44833 | 44944 | 45056 | 45166 | 45277 | 45387 | 45497 | 45607 | 45717 | 11 | 22 | 33 | 44 | 55 |
| 21 | 14491 | 14526 | 14560 | 14595 | 14629 | 14663 | 14697 | 14731 | 14765 | 14799 | 3 | 7 | 10 | 14 | 17 |
| | 45826 | 45935 | 46043 | 46152 | 46260 | 46368 | 46476 | 46583 | 46690 | 46797 | 11 | 22 | 32 | 43 | 54 |
| 22 | 14832 | 14866 | 14900 | 14933 | 14967 | 15000 | 15033 | 15067 | 15100 | 15133 | 3 | 7 | 10 | 13 | 17 |
| | 46904 | 47011 | 47117 | 47223 | 47329 | 47434 | 47539 | 47645 | 47749 | 47854 | 11 | 21 | 32 | 42 | 53 |
| 23 | 15166 | 15199 | 15232 | 15264 | 15297 | 15330 | 15362 | 15395 | 15427 | 15460 | 3 | 7 | 10 | 13 | 16 |
| | 47958 | 48062 | 48166 | 48270 | 48374 | 48477 | 48580 | 48683 | 48785 | 48888 | 10 | 21 | 31 | 41 | 52 |
| 24 | 15492 | 15524 | 15556 | 15588 | 15620 | 15652 | 15684 | 15716 | 15748 | 15780 | 3 | 6 | 10 | 13 | 16 |
| | 48990 | 49092 | 49193 | 49295 | 49396 | 49497 | 49598 | 49699 | 49800 | 49900 | 10 | 20 | 30 | 40 | 51 |
| 25 | 15811 | 15843 | 15875 | 15906 | 15937 | 15969 | 16000 | 16031 | 16062 | 16093 | 3 | 6 | 9 | 13 | 16 |
| | 50000 | 50100 | 50200 | 50299 | 50398 | 50498 | 50596 | 50695 | 50794 | 50892 | 10 | 20 | 30 | 40 | 50 |
| 26 | 16125 | 16155 | 16186 | 16217 | 16248 | 16279 | 16310 | 16340 | 16371 | 16401 | 3 | 6 | 9 | 12 | 15 |
| | 50990 | 51088 | 51186 | 51284 | 51381 | 51478 | 51575 | 51672 | 51769 | 51865 | 10 | 19 | 29 | 39 | 49 |
| 27 | 16432 | 16462 | 16492 | 16523 | 16553 | 16583 | 16613 | 16643 | 16673 | 16703 | 3 | 6 | 9 | 12 | 15 |
| | 51962 | 52058 | 52154 | 52249 | 52345 | 52440 | 52536 | 52631 | 52726 | 52820 | 10 | 19 | 29 | 38 | 48 |
| 28 | 16733 | 16763 | 16793 | 16823 | 16852 | 16882 | 16912 | 16941 | 16971 | 17000 | 3 | 6 | 9 | 12 | 15 |
| | 52915 | 53009 | 53104 | 53198 | 53292 | 53385 | 53479 | 53572 | 53666 | 53759 | 9 | 19 | 28 | 38 | 47 |
| 29 | 17029 | 17059 | 17088 | 17117 | 17146 | 17176 | 17205 | 17234 | 17263 | 17292 | 3 | 6 | 9 | 12 | 15 |
| | 53852 | 53944 | 54037 | 54129 | 54222 | 54314 | 54406 | 54498 | 54589 | 54681 | 9 | 18 | 28 | 37 | 46 |

## TABLE XXVIII. SQUARE ROOTS—*continued*

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 30 | 17321 | 17349 | 17378 | 17407 | 17436 | 17464 | 17493 | 17521 | 17550 | 17578 | 3 | 6 | 9 | 11 | 14 |
|  | 54772 | 54863 | 54955 | 55045 | 55136 | 55227 | 55317 | 55408 | 55498 | 55588 | 9 | 18 | 27 | 36 | 45 |
| 31 | 17607 | 17635 | 17664 | 17692 | 17720 | 17748 | 17776 | 17804 | 17833 | 17861 | 3 | 6 | 8 | 11 | 14 |
|  | 55678 | 55767 | 55857 | 55946 | 56036 | 56125 | 56214 | 56303 | 56391 | 56480 | 9 | 18 | 27 | 36 | 45 |
| 32 | 17889 | 17916 | 17944 | 17972 | 18000 | 18028 | 18055 | 18083 | 18111 | 18138 | 3 | 6 | 8 | 11 | 14 |
|  | 56569 | 56657 | 56745 | 56833 | 56921 | 57009 | 57096 | 57184 | 57271 | 57359 | 9 | 18 | 26 | 35 | 44 |
| 33 | 18166 | 18193 | 18221 | 18248 | 18276 | 18303 | 18330 | 18358 | 18385 | 18412 | 3 | 5 | 8 | 11 | 14 |
|  | 57446 | 57533 | 57619 | 57706 | 57793 | 57879 | 57966 | 58052 | 58138 | 58224 | 9 | 17 | 26 | 35 | 43 |
| 34 | 18439 | 18466 | 18493 | 18520 | 18547 | 18574 | 18601 | 18628 | 18655 | 18682 | 3 | 5 | 8 | 11 | 14 |
|  | 58310 | 58395 | 58481 | 58566 | 58652 | 58737 | 58822 | 58907 | 58992 | 59076 | 9 | 17 | 26 | 34 | 43 |
| 35 | 18708 | 18735 | 18762 | 18788 | 18815 | 18841 | 18868 | 18894 | 18921 | 18947 | 3 | 5 | 8 | 11 | 13 |
|  | 59161 | 59245 | 59330 | 59414 | 59498 | 59582 | 59666 | 59749 | 59833 | 59917 | 8 | 17 | 25 | 34 | 42 |
| 36 | 18974 | 19000 | 19026 | 19053 | 19079 | 19105 | 19131 | 19157 | 19183 | 19209 | 3 | 5 | 8 | 10 | 13 |
|  | 60000 | 60083 | 60166 | 60249 | 60332 | 60415 | 60498 | 60581 | 60663 | 60745 | 8 | 17 | 25 | 33 | 41 |
| 37 | 19235 | 19261 | 19287 | 19313 | 19339 | 19365 | 19391 | 19416 | 19442 | 19468 | 3 | 5 | 8 | 10 | 13 |
|  | 60828 | 60910 | 60992 | 61074 | 61156 | 61237 | 61319 | 61400 | 61482 | 61563 | 8 | 16 | 25 | 33 | 41 |
| 38 | 19494 | 19519 | 19545 | 19570 | 19596 | 19621 | 19647 | 19672 | 19698 | 19723 | 3 | 5 | 8 | 10 | 13 |
|  | 61644 | 61725 | 61806 | 61887 | 61968 | 62048 | 62129 | 62209 | 62290 | 62370 | 8 | 16 | 24 | 32 | 40 |
| 39 | 19748 | 19774 | 19799 | 19824 | 19849 | 19875 | 19900 | 19925 | 19950 | 19975 | 3 | 5 | 8 | 10 | 13 |
|  | 62450 | 62530 | 62610 | 62690 | 62769 | 62849 | 62929 | 63008 | 63087 | 63166 | 8 | 16 | 24 | 32 | 40 |
| 40 | 20000 | 20025 | 20050 | 20075 | 20100 | 20125 | 20149 | 20174 | 20199 | 20224 | 2 | 5 | 7 | 10 | 12 |
|  | 63246 | 63325 | 63403 | 63482 | 63561 | 63640 | 63718 | 63797 | 63875 | 63953 | 8 | 16 | 24 | 31 | 39 |
| 41 | 20248 | 20273 | 20298 | 20322 | 20347 | 20372 | 20396 | 20421 | 20445 | 20469 | 2 | 5 | 7 | 10 | 12 |
|  | 64031 | 64109 | 64187 | 64265 | 64343 | 64420 | 64498 | 64576 | 64653 | 64730 | 8 | 16 | 23 | 31 | 39 |
| 42 | 20494 | 20518 | 20543 | 20567 | 20591 | 20616 | 20640 | 20664 | 20688 | 20712 | 2 | 5 | 7 | 10 | 12 |
|  | 64807 | 64885 | 64962 | 65038 | 65115 | 65192 | 65269 | 65345 | 65422 | 65498 | 8 | 15 | 23 | 31 | 38 |
| 43 | 20736 | 20761 | 20785 | 20809 | 20833 | 20857 | 20881 | 20905 | 20928 | 20952 | 2 | 5 | 7 | 10 | 12 |
|  | 65574 | 65651 | 65727 | 65803 | 65879 | 65955 | 66030 | 66106 | 66182 | 66257 | 8 | 15 | 23 | 30 | 38 |
| 44 | 20976 | 21000 | 21024 | 21048 | 21071 | 21095 | 21119 | 21142 | 21166 | 21190 | 2 | 5 | 7 | 10 | 12 |
|  | 66332 | 66408 | 66483 | 66558 | 66633 | 66708 | 66783 | 66858 | 66933 | 67007 | 8 | 15 | 22 | 30 | 38 |
| 45 | 21213 | 21237 | 21260 | 21284 | 21307 | 21331 | 21354 | 21378 | 21401 | 21424 | 2 | 5 | 7 | 9 | 12 |
|  | 67082 | 67157 | 67231 | 67305 | 67380 | 67454 | 67528 | 67602 | 67676 | 67750 | 7 | 15 | 22 | 30 | 37 |
| 46 | 21448 | 21471 | 21494 | 21517 | 21541 | 21564 | 21587 | 21610 | 21633 | 21656 | 2 | 5 | 7 | 9 | 12 |
|  | 67823 | 67897 | 67971 | 68044 | 68118 | 68191 | 68264 | 68337 | 68411 | 68484 | 7 | 15 | 22 | 29 | 37 |
| 47 | 21679 | 21703 | 21726 | 21749 | 21772 | 21794 | 21817 | 21840 | 21863 | 21886 | 2 | 5 | 7 | 9 | 12 |
|  | 68557 | 68629 | 68702 | 68775 | 68848 | 68920 | 68993 | 69065 | 69138 | 69210 | 7 | 15 | 22 | 29 | 36 |
| 48 | 21909 | 21932 | 21954 | 21977 | 22000 | 22023 | 22045 | 22068 | 22091 | 22113 | 2 | 5 | 7 | 9 | 11 |
|  | 69282 | 69354 | 69426 | 69498 | 69570 | 69642 | 69714 | 69785 | 69857 | 69929 | 7 | 14 | 22 | 29 | 36 |
| 49 | 22136 | 22159 | 22181 | 22204 | 22226 | 22249 | 22271 | 22293 | 22316 | 22338 | 2 | 4 | 7 | 9 | 11 |
|  | 70000 | 70071 | 70143 | 70214 | 70285 | 70356 | 70427 | 70498 | 70569 | 70640 | 7 | 14 | 21 | 28 | 36 |

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 22361 | 22383 | 22405 | 22428 | 22450 | 22472 | 22494 | 22517 | 22539 | 22561 | 2 | 4 | 7 | 9 | 11 |
| | 70711 | 70781 | 70852 | 70922 | 70993 | 71063 | 71134 | 71204 | 71274 | 71344 | 7 | 14 | 21 | 28 | 35 |
| 51 | 22583 | 22605 | 22627 | 22650 | 22672 | 22694 | 22716 | 22738 | 22760 | 22782 | 2 | 4 | 7 | 9 | 11 |
| | 71414 | 71484 | 71554 | 71624 | 71694 | 71764 | 71833 | 71903 | 71972 | 72042 | 7 | 14 | 21 | 28 | 35 |
| 52 | 22804 | 22825 | 22847 | 22869 | 22891 | 22913 | 22935 | 22956 | 22978 | 23000 | 2 | 4 | 7 | 9 | 11 |
| | 72111 | 72180 | 72250 | 72319 | 72388 | 72457 | 72526 | 72595 | 72664 | 72732 | 7 | 14 | 21 | 28 | 34 |
| 53 | 23022 | 23043 | 23065 | 23087 | 23108 | 23130 | 23152 | 23173 | 23195 | 23216 | 2 | 4 | 6 | 9 | 11 |
| | 72801 | 72870 | 72938 | 73007 | 73075 | 73144 | 73212 | 73280 | 73348 | 73417 | 7 | 14 | 21 | 27 | 34 |
| 54 | 23238 | 23259 | 23281 | 23302 | 23324 | 23345 | 23367 | 23388 | 23409 | 23431 | 2 | 4 | 6 | 9 | 11 |
| | 73485 | 73553 | 73621 | 73689 | 73756 | 73824 | 73892 | 73959 | 74027 | 74095 | 7 | 14 | 20 | 27 | 34 |
| 55 | 23452 | 23473 | 23495 | 23516 | 23537 | 23558 | 23580 | 23601 | 23622 | 23643 | 2 | 4 | 6 | 8 | 11 |
| | 74162 | 74229 | 74297 | 74364 | 74431 | 74498 | 74565 | 74632 | 74699 | 74766 | 7 | 13 | 20 | 27 | 34 |
| 56 | 23664 | 23685 | 23707 | 23728 | 23749 | 23770 | 23791 | 23812 | 23833 | 23854 | 2 | 4 | 6 | 8 | 11 |
| | 74833 | 74900 | 74967 | 75033 | 75100 | 75166 | 75233 | 75299 | 75366 | 75432 | 7 | 13 | 20 | 27 | 33 |
| 57 | 23875 | 23896 | 23917 | 23937 | 23958 | 23979 | 24000 | 24021 | 24042 | 24062 | 2 | 4 | 6 | 8 | 10 |
| | 75498 | 75565 | 75631 | 75697 | 75763 | 75829 | 75895 | 75961 | 76026 | 76092 | 7 | 13 | 20 | 26 | 33 |
| 58 | 24083 | 24104 | 24125 | 24145 | 24166 | 24187 | 24207 | 24228 | 24249 | 24269 | 2 | 4 | 6 | 8 | 10 |
| | 76158 | 76223 | 76289 | 76354 | 76420 | 76485 | 76551 | 76616 | 76681 | 76746 | 7 | 13 | 20 | 26 | 33 |
| 59 | 24290 | 24310 | 24331 | 24352 | 24372 | 24393 | 24413 | 24434 | 24454 | 24474 | 2 | 4 | 6 | 8 | 10 |
| | 76811 | 76877 | 76942 | 77006 | 77071 | 77136 | 77201 | 77266 | 77330 | 77395 | 6 | 13 | 19 | 26 | 32 |
| 60 | 24495 | 24515 | 24536 | 24556 | 24576 | 24597 | 24617 | 24637 | 24658 | 24678 | 2 | 4 | 6 | 8 | 10 |
| | 77460 | 77524 | 77589 | 77653 | 77717 | 77782 | 77846 | 77910 | 77974 | 78038 | 6 | 13 | 19 | 26 | 32 |
| 61 | 24698 | 24718 | 24739 | 24759 | 24779 | 24799 | 24819 | 24839 | 24860 | 24880 | 2 | 4 | 6 | 8 | 10 |
| | 78102 | 78166 | 78230 | 78294 | 78358 | 78422 | 78486 | 78549 | 78613 | 78677 | 6 | 13 | 19 | 26 | 32 |
| 62 | 24900 | 24920 | 24940 | 24960 | 24980 | 25000 | 25020 | 25040 | 25060 | 25080 | 2 | 4 | 6 | 8 | 10 |
| | 78740 | 78804 | 78867 | 78930 | 78994 | 79057 | 79120 | 79183 | 79246 | 79310 | 6 | 13 | 19 | 25 | 32 |
| 63 | 25100 | 25120 | 25140 | 25159 | 25179 | 25199 | 25219 | 25239 | 25259 | 25278 | 2 | 4 | 6 | 8 | 10 |
| | 79373 | 79436 | 79498 | 79561 | 79624 | 79687 | 79750 | 79812 | 79875 | 79937 | 6 | 13 | 19 | 25 | 31 |
| 64 | 25298 | 25318 | 25338 | 25357 | 25377 | 25397 | 25417 | 25436 | 25456 | 25475 | 2 | 4 | 6 | 8 | 10 |
| | 80000 | 80062 | 80125 | 80187 | 80250 | 80312 | 80374 | 80436 | 80498 | 80561 | 6 | 12 | 19 | 25 | 31 |
| 65 | 25495 | 25515 | 25534 | 25554 | 25573 | 25593 | 25612 | 25632 | 25652 | 25671 | 2 | 4 | 6 | 8 | 10 |
| | 80623 | 80685 | 80747 | 80808 | 80870 | 80932 | 80994 | 81056 | 81117 | 81179 | 6 | 12 | 19 | 25 | 31 |
| 66 | 25690 | 25710 | 25729 | 25749 | 25768 | 25788 | 25807 | 25826 | 25846 | 25865 | 2 | 4 | 6 | 8 | 10 |
| | 81240 | 81302 | 81363 | 81425 | 81486 | 81548 | 81609 | 81670 | 81731 | 81792 | 6 | 12 | 18 | 25 | 31 |
| 67 | 25884 | 25904 | 25923 | 25942 | 25962 | 25981 | 26000 | 26019 | 26038 | 26058 | 2 | 4 | 6 | 8 | 10 |
| | 81854 | 81915 | 81976 | 82037 | 82098 | 82158 | 82219 | 82280 | 82341 | 82401 | 6 | 12 | 18 | 24 | 30 |
| 68 | 26077 | 26096 | 26115 | 26134 | 26153 | 26173 | 26192 | 26211 | 26230 | 26249 | 2 | 4 | 6 | 8 | 10 |
| | 82462 | 82523 | 82583 | 82644 | 82704 | 82765 | 82825 | 82885 | 82946 | 83006 | 6 | 12 | 18 | 24 | 30 |
| 69 | 26268 | 26287 | 26306 | 26325 | 26344 | 26363 | 26382 | 26401 | 26420 | 26439 | 2 | 4 | 6 | 8 | 10 |
| | 83066 | 83126 | 83187 | 83247 | 83307 | 83367 | 83427 | 83487 | 83546 | 83606 | 6 | 12 | 18 | 24 | 30 |
| 70 | 26458 | 26476 | 26495 | 26514 | 26533 | 26552 | 26571 | 26589 | 26608 | 26627 | 2 | 4 | 6 | 8 | 9 |
| | 83666 | 83726 | 83785 | 83845 | 83905 | 83964 | 84024 | 84083 | 84143 | 84202 | 6 | 12 | 18 | 24 | 30 |
| 71 | 26646 | 26665 | 26683 | 26702 | 26721 | 26739 | 26758 | 26777 | 26796 | 26814 | 2 | 4 | 6 | 7 | 9 |
| | 84261 | 84321 | 84380 | 84439 | 84499 | 84558 | 84617 | 84676 | 84735 | 84794 | 6 | 12 | 18 | 24 | 30 |
| 72 | 26833 | 26851 | 26870 | 26889 | 26907 | 26926 | 26944 | 26963 | 26981 | 27000 | 2 | 4 | 6 | 7 | 9 |
| | 84853 | 84912 | 84971 | 85029 | 85088 | 85147 | 85206 | 85264 | 85323 | 85381 | 6 | 12 | 18 | 23 | 29 |
| 73 | 27019 | 27037 | 27055 | 27074 | 27092 | 27111 | 27129 | 27148 | 27166 | 27185 | 2 | 4 | 6 | 7 | 9 |
| | 85440 | 85499 | 85557 | 85615 | 85674 | 85732 | 85790 | 85849 | 85907 | 85965 | 6 | 12 | 17 | 23 | 29 |
| 74 | 27203 | 27221 | 27240 | 27258 | 27276 | 27295 | 27313 | 27331 | 27350 | 27368 | 2 | 4 | 5 | 7 | 9 |
| | 86023 | 86081 | 86139 | 86197 | 86255 | 86313 | 86371 | 86429 | 86487 | 86545 | 6 | 12 | 17 | 23 | 29 |

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 75 | 27386 | 27404 | 27423 | 27441 | 27459 | 27477 | 27495 | 27514 | 27532 | 27550 | 2 | 4 | 5 | 7 | 9 |
|  | 86603 | 86660 | 86718 | 86776 | 86833 | 86891 | 86948 | 87006 | 87063 | 87121 | 6 | 12 | 17 | 23 | 29 |
| 76 | 27568 | 27586 | 27604 | 27622 | 27641 | 27659 | 27677 | 27695 | 27713 | 27731 | 2 | 4 | 5 | 7 | 9 |
|  | 87178 | 87235 | 87293 | 87350 | 87407 | 87464 | 87521 | 87579 | 87636 | 87693 | 6 | 11 | 17 | 23 | 29 |
| 77 | 27749 | 27767 | 27785 | 27803 | 27821 | 27839 | 27857 | 27875 | 27893 | 27911 | 2 | 4 | 5 | 7 | 9 |
|  | 87750 | 87807 | 87864 | 87920 | 87977 | 88034 | 88091 | 88148 | 88204 | 88261 | 6 | 11 | 17 | 23 | 28 |
| 78 | 27928 | 27946 | 27964 | 27982 | 28000 | 28018 | 28036 | 28054 | 28071 | 28089 | 2 | 4 | 5 | 7 | 9 |
|  | 88318 | 88374 | 88431 | 88487 | 88544 | 88600 | 88657 | 88713 | 88769 | 88826 | 6 | 11 | 17 | 23 | 28 |
| 79 | 28107 | 28125 | 28142 | 28160 | 28178 | 28196 | 28213 | 28231 | 28249 | 28267 | 2 | 4 | 5 | 7 | 9 |
|  | 88882 | 88938 | 88994 | 89051 | 89107 | 89163 | 89219 | 89275 | 89331 | 89387 | 6 | 11 | 17 | 22 | 28 |
| 80 | 28284 | 28302 | 28320 | 28337 | 28355 | 28373 | 28390 | 28408 | 28425 | 28443 | 2 | 4 | 5 | 7 | 9 |
|  | 89443 | 89499 | 89554 | 89610 | 89666 | 89722 | 89778 | 89833 | 89889 | 89944 | 6 | 11 | 17 | 22 | 28 |
| 81 | 28460 | 28478 | 28496 | 28513 | 28531 | 28548 | 28566 | 28583 | 28601 | 28618 | 2 | 4 | 5 | 7 | 9 |
|  | 90000 | 90056 | 90111 | 90167 | 90222 | 90277 | 90333 | 90388 | 90443 | 90499 | 6 | 11 | 17 | 22 | 28 |
| 82 | 28636 | 28653 | 28671 | 28688 | 28705 | 28723 | 28740 | 28758 | 28775 | 28792 | 2 | 3 | 5 | 7 | 9 |
|  | 90554 | 90609 | 90664 | 90719 | 90774 | 90830 | 90885 | 90940 | 90995 | 91049 | 6 | 11 | 16 | 22 | 28 |
| 83 | 28810 | 28827 | 28844 | 28862 | 28879 | 28896 | 28914 | 28931 | 28948 | 28965 | 2 | 3 | 5 | 7 | 9 |
|  | 91104 | 91159 | 91214 | 91269 | 91324 | 91378 | 91433 | 91488 | 91542 | 91597 | 5 | 11 | 16 | 22 | 27 |
| 84 | 28983 | 29000 | 29017 | 29034 | 29052 | 29069 | 29086 | 29103 | 29120 | 29138 | 2 | 3 | 5 | 7 | 9 |
|  | 91652 | 91706 | 91761 | 91815 | 91869 | 91924 | 91978 | 92033 | 92087 | 92141 | 5 | 11 | 16 | 22 | 27 |
| 85 | 29155 | 29172 | 29189 | 29206 | 29223 | 29240 | 29257 | 29275 | 29292 | 29309 | 2 | 3 | 5 | 7 | 9 |
|  | 92195 | 92250 | 92304 | 92358 | 92412 | 92466 | 92520 | 92574 | 92628 | 92682 | 5 | 11 | 16 | 22 | 27 |
| 86 | 29326 | 29343 | 29360 | 29377 | 29394 | 29411 | 29428 | 29445 | 29462 | 29479 | 2 | 3 | 5 | 7 | 8 |
|  | 92736 | 92790 | 92844 | 92898 | 92952 | 93005 | 93059 | 93113 | 93167 | 93220 | 5 | 11 | 16 | 22 | 27 |
| 87 | 29496 | 29513 | 29530 | 29547 | 29563 | 29580 | 29597 | 29614 | 29631 | 29648 | 2 | 3 | 5 | 7 | 8 |
|  | 93274 | 93327 | 93381 | 93434 | 93488 | 93541 | 93595 | 93648 | 93702 | 93755 | 5 | 11 | 16 | 21 | 27 |
| 88 | 29665 | 29682 | 29698 | 29715 | 29732 | 29749 | 29766 | 29783 | 29799 | 29816 | 2 | 3 | 5 | 7 | 8 |
|  | 93808 | 93862 | 93915 | 93968 | 94021 | 94074 | 94128 | 94181 | 94234 | 94287 | 5 | 11 | 16 | 21 | 27 |
| 89 | 29833 | 29850 | 29866 | 29883 | 29900 | 29917 | 29933 | 29950 | 29967 | 29983 | 2 | 3 | 5 | 7 | 8 |
|  | 94340 | 94393 | 94446 | 94499 | 94552 | 94604 | 94657 | 94710 | 94763 | 94816 | 5 | 11 | 16 | 21 | 26 |
| 90 | 30000 | 30017 | 30033 | 30050 | 30067 | 30083 | 30100 | 30116 | 30133 | 30150 | 2 | 3 | 5 | 7 | 8 |
|  | 94868 | 94921 | 94974 | 95026 | 95079 | 95131 | 95184 | 95237 | 95289 | 95341 | 5 | 11 | 16 | 21 | 26 |
| 91 | 30166 | 30183 | 30199 | 30216 | 30232 | 30249 | 30265 | 30282 | 30299 | 30315 | 2 | 3 | 5 | 7 | 8 |
|  | 95394 | 95446 | 95499 | 95551 | 95603 | 95656 | 95708 | 95760 | 95812 | 95864 | 5 | 10 | 16 | 21 | 26 |
| 92 | 30332 | 30348 | 30364 | 30381 | 30397 | 30414 | 30430 | 30447 | 30463 | 30480 | 2 | 3 | 5 | 7 | 8 |
|  | 95917 | 95969 | 96021 | 96073 | 96125 | 96177 | 96229 | 96281 | 96333 | 96385 | 5 | 10 | 16 | 21 | 26 |
| 93 | 30496 | 30512 | 30529 | 30545 | 30561 | 30578 | 30594 | 30610 | 30627 | 30643 | 2 | 3 | 5 | 7 | 8 |
|  | 96437 | 96488 | 96540 | 96592 | 96644 | 96695 | 96747 | 96799 | 96850 | 96902 | 5 | 10 | 16 | 21 | 26 |
| 94 | 30659 | 30676 | 30692 | 30708 | 30725 | 30741 | 30757 | 30773 | 30790 | 30806 | 2 | 3 | 5 | 7 | 8 |
|  | 96954 | 97005 | 97057 | 97108 | 97160 | 97211 | 97263 | 97314 | 97365 | 97417 | 5 | 10 | 15 | 21 | 26 |
| 95 | 30822 | 30838 | 30854 | 30871 | 30887 | 30903 | 30919 | 30935 | 30952 | 30968 | 2 | 3 | 5 | 6 | 8 |
|  | 97468 | 97519 | 97570 | 97622 | 97673 | 97724 | 97775 | 97826 | 97877 | 97929 | 5 | 10 | 15 | 20 | 26 |
| 96 | 30984 | 31000 | 31016 | 31032 | 31048 | 31064 | 31081 | 31097 | 31113 | 31129 | 2 | 3 | 5 | 6 | 8 |
|  | 97980 | 98031 | 98082 | 98133 | 98184 | 98234 | 98285 | 98336 | 98387 | 98438 | 5 | 10 | 15 | 20 | 25 |
| 97 | 31145 | 31161 | 31177 | 31193 | 31209 | 31225 | 31241 | 31257 | 31273 | 31289 | 2 | 3 | 5 | 6 | 8 |
|  | 98489 | 98539 | 98590 | 98641 | 98691 | 98742 | 98793 | 98843 | 98894 | 98944 | 5 | 10 | 15 | 20 | 25 |
| 98 | 31305 | 31321 | 31337 | 31353 | 31369 | 31385 | 31401 | 31417 | 31432 | 31448 | 2 | 3 | 5 | 6 | 8 |
|  | 98995 | 99045 | 99096 | 99146 | 99197 | 99247 | 99298 | 99348 | 99398 | 99448 | 5 | 10 | 15 | 20 | 25 |
| 99 | 31464 | 31480 | 31496 | 31512 | 31528 | 31544 | 31559 | 31575 | 31591 | 31607 | 2 | 3 | 5 | 6 | 8 |
|  | 99499 | 99549 | 99599 | 99649 | 99700 | 99750 | 99800 | 99850 | 99900 | 99950 | 5 | 10 | 15 | 20 | 25 |

# TABLE XXIX. RECIPROCALS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Diffs.* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1·0 | 1·000000 | 990099 | 980392 | 970874 | 961538 | 952381 | 943396 | 934579 | 925926 | 917431 | |
| | −9901 | −9707 | −9518 | −9336 | −9157 | −8985 | −8817 | −8653 | −8495 | −8340 | |
| 1·1 | ·909091 | 900901 | 892857 | 884956 | 877193 | 869565 | 862069 | 854701 | 847458 | 840336 | |
| | −8190 | −8044 | −7901 | −7763 | −7628 | −7496 | −7368 | −7243 | −7122 | −7003 | |
| 1·2 | ·833333 | 826446 | 819672 | 813008 | 806452 | 800000 | 793651 | 787402 | 781250 | 775194 | |
| | −6887 | −6774 | −6664 | −6556 | −6452 | −6349 | −6249 | −6152 | −6056 | −5963 | |
| 1·3 | ·769231 | 763359 | 757576 | 751880 | 746269 | 740741 | 735294 | 729927 | 724638 | 719424 | |
| | −5872 | −5783 | −5696 | −5611 | −5528 | −5447 | −5367 | −5289 | −5214 | −5138 | |
| 1·4 | ·714286 | 709220 | 704225 | 699301 | 694444 | 689655 | 684932 | 680272 | 675676 | 671141 | |
| | −5066 | −4995 | −4924 | −4857 | −4789 | −4723 | −4660 | −4596 | −4535 | −4474 | |
| 1·5 | ·666667 | 662252 | 657895 | 653595 | 649351 | 645161 | 641026 | 636943 | 632911 | 628931 | |
| | −4415 | −4357 | −4300 | −4244 | −4190 | −4135 | −4083 | −4032 | −3980 | −3931 | |
| 1·6 | ·625000 | 621118 | 617284 | 613497 | 609756 | 606061 | 602410 | 598802 | 595238 | 591716 | |
| | −3882 | −3834 | −3787 | −3741 | −3695 | −3651 | −3608 | −3564 | −3522 | −3481 | |
| 1·7 | ·588235 | 584795 | 581395 | 578035 | 574713 | 571429 | 568182 | 564972 | 561798 | 558659 | |
| | −3440 | −3400 | −3360 | −3322 | −3284 | −3247 | −3210 | −3174 | −3139 | −3103 | |
| 1·8 | ·555556 | 552486 | 549451 | 546448 | 543478 | 540541 | 537634 | 534759 | 531915 | 529101 | |
| | −3070 | −3035 | −3003 | −2970 | −2937 | −2907 | −2875 | −2844 | −2814 | −2785 | |
| 1·9 | ·526316 | 523560 | 520833 | 518135 | 515464 | 512821 | 510204 | 507614 | 505051 | 502513 | |
| | −2756 | −2727 | −2698 | −2671 | −2643 | −2617 | −2590 | −2563 | −2538 | −2513 | |
| 2·0 | ·500000 | 497512 | 495050 | 492611 | 490196 | 487805 | 485437 | 483092 | 480769 | 478469 | |
| | −2488 | −2462 | −2439 | −2415 | −2391 | −2368 | −2345 | −2323 | −2300 | −2279 | |
| 2·1 | ·476190 | 473934 | 471698 | 469484 | 467290 | 465116 | 462963 | 460829 | 458716 | 456621 | |
| | −2256 | −2236 | −2214 | −2194 | −2174 | −2153 | −2134 | −2113 | −2095 | −2076 | |
| 2·2 | ·454545 | 452489 | 450450 | 448430 | 446429 | 444444 | 442478 | 440529 | 438596 | 436681 | |
| | −2056 | −2039 | −2020 | −2001 | −1985 | −1966 | −1949 | −1933 | −1915 | −1898 | |
| 2·3 | ·434783 | 432900 | 431034 | 429185 | 427350 | 425532 | 423729 | 421941 | 420168 | 418410 | |
| | −1883 | −1866 | −1849 | −1835 | −1818 | −1803 | −1788 | −1773 | −1758 | −1743 | |
| 2·4 | ·416667 | 414938 | 413223 | 411523 | 409836 | 408163 | 406504 | 404858 | 403226 | 401606 | |
| | −1729 | −1715 | −1700 | −1687 | −1673 | −1659 | −1646 | −1632 | −1620 | −1606 | |
| 2·5 | ·400000 | 398406 | 396825 | 395257 | 393701 | 392157 | 390625 | 389105 | 387597 | 386100 | −1544 |
| 2·6 | ·384615 | 383142 | 381679 | 380228 | 378788 | 377358 | 375940 | 374532 | 373134 | 371747 | −1430 |
| 2·7 | ·370370 | 369004 | 367647 | 366300 | 364964 | 363636 | 362319 | 361011 | 359712 | 358423 | −1327 |
| 2·8 | ·357143 | 355872 | 354610 | 353357 | 352113 | 350877 | 349650 | 348432 | 347222 | 346021 | −1236 |
| 2·9 | ·344828 | 343643 | 342466 | 341297 | 340136 | 338983 | 337838 | 336700 | 335570 | 334448 | −1153 |
| 3·0 | ·333333 | 332226 | 331126 | 330033 | 328947 | 327869 | 326797 | 325733 | 324675 | 323625 | −1079 |
| 3·1 | ·322581 | 321543 | 320513 | 319489 | 318471 | 317460 | 316456 | 315457 | 314465 | 313480 | −1011 |
| 3·2 | ·312500 | 311526 | 310559 | 309598 | 308642 | 307692 | 306748 | 305810 | 304878 | 303951 | −950 |
| 3·3 | ·303030 | 302115 | 301205 | 300300 | 299401 | 298507 | 297619 | 296736 | 295858 | 294985 | −894 |
| 3·4 | ·294118 | 293255 | 292398 | 291545 | 290698 | 289855 | 289017 | 288184 | 287356 | 286533 | −843 |
| 3·5 | ·285714 | 284900 | 284091 | 283286 | 282486 | 281690 | 280899 | 280112 | 279330 | 278552 | −796 |
| 3·6 | ·277778 | 277008 | 276243 | 275482 | 274725 | 273973 | 273224 | 272480 | 271739 | 271003 | −753 |
| 3·7 | ·270270 | 269542 | 268817 | 268097 | 267380 | 266667 | 265957 | 265252 | 264550 | 263852 | −713 |
| 3·8 | ·263158 | 262467 | 261780 | 261097 | 260417 | 259740 | 259067 | 258398 | 257732 | 257069 | −677 |
| 3·9 | ·256410 | 255754 | 255102 | 254453 | 253807 | 253165 | 252525 | 251889 | 251256 | 250627 | −643 |
| 4·0 | ·250000 | 249377 | 248756 | 248139 | 247525 | 246914 | 246305 | 245700 | 245098 | 244499 | −611 |
| 4·1 | ·243902 | 243309 | 242718 | 242131 | 241546 | 240964 | 240385 | 239808 | 239234 | 238663 | −582 |
| 4·2 | ·238095 | 237530 | 236967 | 236407 | 235849 | 235294 | 234742 | 234192 | 233645 | 233100 | −555 |
| 4·3 | ·232558 | 232019 | 231481 | 230947 | 230415 | 229885 | 229358 | 228833 | 228311 | 227790 | −530 |
| 4·4 | ·227273 | 226757 | 226244 | 225734 | 225225 | 224719 | 224215 | 223714 | 223214 | 222717 | −506 |
| 4·5 | ·222222 | 221729 | 221239 | 220751 | 220264 | 219780 | 219298 | 218818 | 218341 | 217865 | −484 |
| 4·6 | ·217391 | 216920 | 216450 | 215983 | 215517 | 215054 | 214592 | 214133 | 213675 | 213220 | −463 |
| 4·7 | ·212766 | 212314 | 211864 | 211416 | 210970 | 210526 | 210084 | 209644 | 209205 | 208768 | −444 |
| 4·8 | ·208333 | 207900 | 207469 | 207039 | 206612 | 206186 | 205761 | 205339 | 204918 | 204499 | −426 |
| 4·9 | ·204082 | 203666 | 203252 | 202840 | 202429 | 202020 | 201613 | 201207 | 200803 | 200401 | −409 |

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* Tabular differences up to 2·5, and mean differences thereafter. Differences are *negative*.

## TABLE XXIX. RECIPROCALS—*continued*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Diffs.* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5·0 | ·200000 | 199601 | 199203 | 198807 | 198413 | 198020 | 197628 | 197239 | 196850 | 196464 | −393 |
| 5·1 | ·196078 | 195695 | 195312 | 194932 | 194553 | 194175 | 193798 | 193424 | 193050 | 192678 | −378 |
| 5·2 | ·192308 | 191939 | 191571 | 191205 | 190840 | 190476 | 190114 | 189753 | 189394 | 189036 | −364 |
| 5·3 | ·188679 | 188324 | 187970 | 187617 | 187266 | 186916 | 186567 | 186220 | 185874 | 185529 | −350 |
| 5·4 | ·185185 | 184843 | 184502 | 184162 | 183824 | 183486 | 183150 | 182815 | 182482 | 182149 | −337 |
| 5·5 | ·181818 | 181488 | 181159 | 180832 | 180505 | 180180 | 179856 | 179533 | 179211 | 178891 | -325 |
| 5·6 | ·178571 | 178253 | 177936 | 177620 | 177305 | 176991 | 176678 | 176367 | 176056 | 175747 | 314 |
| 5·7 | ·175439 | 175131 | 174825 | 174520 | 174216 | 173913 | 173611 | 173310 | 173010 | 172712 | −303 |
| 5·8 | ·172414 | 172117 | 171821 | 171527 | 171233 | 170940 | 170648 | 170358 | 170068 | 169779 | −293 |
| 5·9 | ·169492 | 169205 | 168919 | 168634 | 168350 | 168067 | 167785 | 167504 | 167224 | 166945 | −283 |
| 6·0 | ·166667 | 166389 | 166113 | 165837 | 165563 | 165289 | 165017 | 164745 | 164474 | 164204 | −274 |
| 6·1 | ·163934 | 163666 | 163399 | 163132 | 162866 | 162602 | 162338 | 162075 | 161812 | 161551 | −265 |
| 6·2 | ·161290 | 161031 | 160772 | 160514 | 160256 | 160000 | 159744 | 159490 | 159236 | 158983 | −256 |
| 6·3 | ·158730 | 158479 | 158228 | 157978 | 157729 | 157480 | 157233 | 156986 | 156740 | 156495 | −248 |
| 6·4 | ·156250 | 156006 | 155763 | 155521 | 155280 | 155039 | 154799 | 154560 | 154321 | 154083 | −241 |
| 6·5 | ·153846 | 153610 | 153374 | 153139 | 152905 | 152672 | 152439 | 152207 | 151976 | 151745 | −233 |
| 6·6 | ·151515 | 151286 | 151057 | 150830 | 150602 | 150376 | 150150 | 149925 | 149701 | 149477 | −226 |
| 6·7 | ·149254 | 149031 | 148810 | 148588 | 148368 | 148148 | 147929 | 147710 | 147493 | 147275 | −220 |
| 6·8 | ·147059 | 146843 | 146628 | 146413 | 146199 | 145985 | 145773 | 145560 | 145349 | 145138 | −213 |
| 6·9 | ·144928 | 144718 | 144509 | 144300 | 144092 | 143885 | 143678 | 143472 | 143266 | 143062 | −207 |
| 7·0 | ·142857 | 142653 | 142450 | 142248 | 142045 | 141844 | 141643 | 141443 | 141243 | 141044 | −201 |
| 7·1 | ·140845 | 140647 | 140449 | 140252 | 140056 | 139860 | 139665 | 139470 | 139276 | 139082 | −196 |
| 7·2 | ·138889 | 138696 | 138504 | 138313 | 138122 | 137931 | 137741 | 137552 | 137363 | 137174 | −191 |
| 7·3 | ·136986 | 136799 | 136612 | 136426 | 136240 | 136054 | 135870 | 135685 | 135501 | 135318 | −185 |
| 7·4 | ·135135 | 134953 | 134771 | 134590 | 134409 | 134228 | 134048 | 133869 | 133690 | 133511 | −180 |
| 7·5 | ·133333 | 133156 | 132979 | 132802 | 132626 | 132450 | 132275 | 132100 | 131926 | 131752 | −176 |
| 7·6 | ·131579 | 131406 | 131234 | 131062 | 130890 | 130719 | 130548 | 130378 | 130208 | 130039 | −172 |
| 7·7 | ·129870 | 129702 | 129534 | 129366 | 129199 | 129032 | 128866 | 128700 | 128535 | 128370 | −167 |
| 7·8 | ·128205 | 128041 | 127877 | 127714 | 127551 | 127389 | 127226 | 127065 | 126904 | 126743 | −162 |
| 7·9 | ·126582 | 126422 | 126263 | 126103 | 125945 | 125786 | 125628 | 125471 | 125313 | 125156 | −158 |
| 8·0 | ·125000 | 124844 | 124688 | 124533 | 124378 | 124224 | 124069 | 123916 | 123762 | 123609 | −155 |
| 8·1 | ·123457 | 123305 | 123153 | 123001 | 122850 | 122699 | 122549 | 122399 | 122249 | 122100 | −151 |
| 8·2 | ·121951 | 121803 | 121655 | 121507 | 121359 | 121212 | 121065 | 120919 | 120773 | 120627 | −147 |
| 8·3 | ·120482 | 120337 | 120192 | 120048 | 119904 | 119760 | 119617 | 119474 | 119332 | 119190 | −144 |
| 8·4 | ·119048 | 118906 | 118765 | 118624 | 118483 | 118343 | 118203 | 118064 | 117925 | 117786 | −140 |
| 8·5 | ·117647 | 117509 | 117371 | 117233 | 117096 | 116959 | 116822 | 116686 | 116550 | 116414 | −137 |
| 8·6 | ·116279 | 116144 | 116009 | 115875 | 115741 | 115607 | 115473 | 115340 | 115207 | 115075 | −134 |
| 8·7 | ·114943 | 114811 | 114679 | 114548 | 114416 | 114286 | 114155 | 114025 | 113895 | 113766 | −131 |
| 8·8 | ·113636 | 113507 | 113379 | 113250 | 113122 | 112994 | 112867 | 112740 | 112613 | 112486 | −128 |
| 8·9 | ·112360 | 112233 | 112108 | 111982 | 111857 | 111732 | 111607 | 111483 | 111359 | 111235 | −125 |
| 9·0 | ·111111 | 110988 | 110865 | 110742 | 110619 | 110497 | 110375 | 110254 | 110132 | 110011 | −122 |
| 9·1 | ·109890 | 109769 | 109649 | 109529 | 109409 | 109290 | 109170 | 109051 | 108932 | 108814 | −120 |
| 9·2 | ·108696 | 108578 | 108460 | 108342 | 108225 | 108108 | 107991 | 107875 | 107759 | 107643 | −117 |
| 9·3 | ·107527 | 107411 | 107296 | 107181 | 107066 | 106952 | 106838 | 106724 | 106610 | 106496 | −115 |
| 9·4 | ·106383 | 106270 | 106157 | 106045 | 105932 | 105820 | 105708 | 105597 | 105485 | 105374 | −112 |
| 9·5 | ·105263 | 105152 | 105042 | 104932 | 104822 | 104712 | 104603 | 104493 | 104384 | 104275 | −110 |
| 9·6 | ·104167 | 104058 | 103950 | 103842 | 103734 | 103627 | 103520 | 103413 | 103306 | 103199 | −108 |
| 9·7 | ·103093 | 102987 | 102881 | 102775 | 102669 | 102564 | 102459 | 102354 | 102249 | 102145 | −105 |
| 9·8 | ·102041 | 101937 | 101833 | 101729 | 101626 | 101523 | 101420 | 101317 | 101215 | 101112 | −103 |
| 9·9 | ·101010 | 100908 | 100806 | 100705 | 100604 | 100503 | 100402 | 100301 | 100200 | 100100 | −101 |

Thus 1/4·214 = 0·237530−0·000222 = 0·237308, since 0·4 × 555 = 222 (correct value 0·237304).

# TABLE XXX. FACTORIALS

| No. | Factorial. | Logarithm. | No. | Factorial. | Logarithm. | No. | Factorial. | Logarithm. |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0·000 0000 | 51 | 1·55112 | 66·190 6450 | 101 | 9·42595 | 159·974 3250 |
| 2 | 2 | 0·301 0300 | 52 | 8·06582 | 67·906 6484 | 102 | 9·61447 | 161·982 9252 |
| 3 | 6 | 0·778 1513 | 53 | 4·27488 | 69·630 9243 | 103 | 9·90290 | 163·995 7624 |
| 4 | 24 | 1·380 2112 | 54 | 2·30844 | 71·363 3180 | 104 | 1·02990 | 166·012 7958 |
| 5 | 120 | 2·079 1812 | 55 | 1·26964 | 73·103 6807 | 105 | 1·08140 | 168·033 9851 |
| 6 | 720 | 2·857 3325 | 56 | 7·10999 | 74·851 8687 | 106 | 1·14628 | 170·059 2909 |
| 7 | 5040 | 3·702 4305 | 57 | 4·05269 | 76·607 7436 | 107 | 1·22652 | 172·088 6747 |
| 8 | 40320 | 4·605 5205 | 58 | 2·35056 | 78·371 1716 | 108 | 1·32464 | 174·122 0985 |
| 9 | 362880 | 5·559 7630 | 59 | 1·38683 | 80·142 0236 | 109 | 1·44386 | 176·159 5250 |
| 10 | 3·62880 | 6·559 7630 | 60 | 8·32099 | 81·920 1748 | 110 | 1·58825 | 178·200 9176 |
| 11 | 3·99168 | 7·601 1557 | 61 | 5·07580 | 83·705 5047 | 111 | 1·76295 | 180·246 2406 |
| 12 | 4·79002 | 8·680 3370 | 62 | 3·14700 | 85·497 8964 | 112 | 1·97451 | 182·295 4586 |
| 13 | 6·22702 | 9·794 2803 | 63 | 1·98261 | 87·297 2369 | 113 | 2·23119 | 184·348 5371 |
| 14 | 8·71783 | 10·940 4084 | 64 | 1·26887 | 89·103 4169 | 114 | 2·54356 | 186·405 4419 |
| 15 | 1·30767 | 12·116 4996 | 65 | 8·24765 | 90·916 3303 | 115 | 2·92509 | 188·466 1398 |
| 16 | 2·09228 | 13·320 6196 | 66 | 5·44345 | 92·735 8742 | 116 | 3·39311 | 190·530 5978 |
| 17 | 3·55687 | 14·551 0685 | 67 | 3·64711 | 94·561 9490 | 117 | 3·96994 | 192·598 7836 |
| 18 | 6·40237 | 15·806 3410 | 68 | 2·48004 | 96·394 4579 | 118 | 4·68453 | 194·670 6656 |
| 19 | 1·21645 | 17·085 0946 | 69 | 1·71122 | 98·233 3070 | 119 | 5·57459 | 196·746 2126 |
| 20 | 2·43290 | 18·386 1246 | 70 | 1·19786 | 100·078 4050 | 120 | 6·68950 | 198·825 3938 |
| 21 | 5·10909 | 19·708 3439 | 71 | 8·50479 | 101·929 6634 | 121 | 8·09430 | 200·908 1792 |
| 22 | 1·12400 | 21·050 7666 | 72 | 6·12345 | 103·786 9959 | 122 | 9·87504 | 202·994 5390 |
| 23 | 2·58520 | 22·412 4944 | 73 | 4·47012 | 105·650 3187 | 123 | 1·21463 | 205·084 4442 |
| 24 | 6·20448 | 23·792 7057 | 74 | 3·30789 | 107·519 5505 | 124 | 1·50614 | 207·177 8658 |
| 25 | 1·55112 | 25·190 6457 | 75 | 2·48091 | 109·394 6117 | 125 | 1·88268 | 209·274 7759 |
| 26 | 4·03291 | 26·605 6190 | 76 | 1·88549 | 111·275 4253 | 126 | 2·37217 | 211·375 1464 |
| 27 | 1·08889 | 28·036 9828 | 77 | 1·45183 | 113·161 9160 | 127 | 3·01266 | 213·478 9501 |
| 28 | 3·04888 | 29·484 1408 | 78 | 1·13243 | 115·054 0106 | 128 | 3·85620 | 215·586 1601 |
| 29 | 8·84176 | 30·946 5388 | 79 | 8·94618 | 116·951 6377 | 129 | 4·97450 | 217·696 7498 |
| 30 | 2·65253 | 32·423 6601 | 80 | 7·15695 | 118·854 7277 | 130 | 6·46686 | 219·810 6932 |
| 31 | 8·22284 | 33·915 0218 | 81 | 5·79713 | 120·763 2127 | 131 | 8·47158 | 221·927 9645 |
| 32 | 2·63131 | 35·420 1717 | 82 | 4·75364 | 122·677 0266 | 132 | 1·11825 | 224·048 5384 |
| 33 | 8·68332 | 36·938 6857 | 83 | 3·94552 | 124·596 1047 | 133 | 1·48727 | 226·172 3900 |
| 34 | 2·95233 | 38·470 1646 | 84 | 3·31424 | 126·520 3840 | 134 | 1·99294 | 228·299 4948 |
| 35 | 1·03331 | 40·014 2326 | 85 | 2·81710 | 128·449 8029 | 135 | 2·69047 | 230·429 8286 |
| 36 | 3·71993 | 41·570 5351 | 86 | 2·42271 | 130·384 3013 | 136 | 3·65904 | 232·563 3675 |
| 37 | 1·37638 | 43·138 7369 | 87 | 2·10776 | 132·323 8206 | 137 | 5·01289 | 234·700 0881 |
| 38 | 5·23023 | 44·718 5205 | 88 | 1·85483 | 134·268 3033 | 138 | 6·91779 | 236·839 9672 |
| 39 | 2·03979 | 46·309 5851 | 89 | 1·65080 | 136·217 6933 | 139 | 9·61572 | 238·982 9820 |
| 40 | 8·15915 | 47·911 6451 | 90 | 1·48572 | 138·171 9358 | 140 | 1·34620 | 241·129 1100 |
| 41 | 3·34525 | 49·524 4289 | 91 | 1·35200 | 140·130 9772 | 141 | 1·89814 | 243·278 3291 |
| 42 | 1·40501 | 51·147 6782 | 92 | 1·24384 | 142·094 7650 | 142 | 2·69536 | 245·430 6174 |
| 43 | 6·04153 | 52·781 1467 | 93 | 1·15677 | 144·063 2480 | 143 | 3·85437 | 247·585 9535 |
| 44 | 2·65827 | 54·424 5993 | 94 | 1·08737 | 146·036 3758 | 144 | 5·55029 | 249·744 3160 |
| 45 | 1·19622 | 56·077 8119 | 95 | 1·03300 | 148·014 0994 | 145 | 8·04793 | 251·905 6840 |
| 46 | 5·50262 | 57·740 5697 | 96 | 9·91678 | 149·996 3707 | 146 | 1·17500 | 254·070 0368 |
| 47 | 2·58623 | 59·412 6676 | 97 | 9·61928 | 151·983 1424 | 147 | 1·72725 | 256·237 3542 |
| 48 | 1·24139 | 61·093 9088 | 98 | 9·42689 | 153·974 3685 | 148 | 2·55632 | 258·407 6159 |
| 49 | 6·08282 | 62·784 1049 | 99 | 9·33262 | 155·970 0037 | 149 | 3·80892 | 260·580 8022 |
| 50 | 3·04141 | 64·483 0749 | 100 | 9·33262 | 157·970 0037 | 150 | 5·71338 | 262·756 8934 |

The power of 10 by which to multiply the factorial is given by the whole number of the logarithm.

# TABLE XXX. FACTORIALS—*continued*

| No. | Factorial. | Logarithm. | No. | Factorial. | Logarithm. | No. | Factorial. | Logarithm. |
|---|---|---|---|---|---|---|---|---|
| 151 | 8·62721 | 264·935 8704 | 201 | 1·58520 | 377·200 0847 | 251 | 8·11447 | 494·909 2601 |
| 152 | 1·31134 | 267·117 7139 | 202 | 3·20211 | 379·505 4361 | 252 | 2·04485 | 497·310 6607 |
| 153 | 2·00634 | 269·302 4054 | 203 | 6·50028 | 381·812 9321 | 253 | 5·17346 | 499·713 7812 |
| 154 | 3·08977 | 271·489 9261 | 204 | 1·32606 | 384·122 5623 | 254 | 1·31406 | 502·118 6149 |
| 155 | 4·78914 | 273·680 2578 | 205 | 2·71842 | 386·434 3161 | 255 | 3·35085 | 504·525 1551 |
| 156 | 7·47106 | 275·873 3824 | 206 | 5·59994 | 388·748 1834 | 256 | 8·57818 | 506·933 3950 |
| 157 | 1·17296 | 278·069 2820 | 207 | 1·15919 | 391·064 1537 | 257 | 2·20459 | 509·343 3282 |
| 158 | 1·85327 | 280·267 9391 | 208 | 2·41111 | 393·382 2170 | 258 | 5·68785 | 511·754 9479 |
| 159 | 2·94670 | 282·469 3363 | 209 | 5·03922 | 395·702 3633 | 259 | 1·47315 | 514·168 2476 |
| 160 | 4·71472 | 284·673 4562 | 210 | 1·05824 | 398·024 5826 | 260 | 3·83020 | 516·583 2210 |
| 161 | 7·59070 | 286·880 2821 | 211 | 2·23288 | 400·348 8651 | 261 | 9·99681 | 518·999 8615 |
| 162 | 1·22969 | 289·089 7971 | 212 | 4·73370 | 402·675 2009 | 262 | 2·61916 | 521·418 1628 |
| 163 | 2·00440 | 291·301 9847 | 213 | 1·00828 | 405·003 5805 | 263 | 6·88840 | 523·838 1185 |
| 164 | 3·28722 | 293·516 8286 | 214 | 2·15772 | 407·333 9943 | 264 | 1·81854 | 526·259 7225 |
| 165 | 5·42391 | 295·734 3125 | 215 | 4·63909 | 409·666 4328 | 265 | 4·81913 | 528·682 9683 |
| 166 | 9·00369 | 297·954 4206 | 216 | 1·00204 | 412·000 8865 | 266 | 1·28189 | 531·107 8500 |
| 167 | 1·50362 | 300·177 1371 | 217 | 2·17443 | 414·337 3463 | 267 | 3·42264 | 533·534 3612 |
| 168 | 2·52608 | 302·402 4464 | 218 | 4·74027 | 416·675 8027 | 268 | 9·17268 | 535·962 4960 |
| 169 | 4·26907 | 304·630 3331 | 219 | 1·03812 | 419·016 2469 | 269 | 2·46745 | 538·392 2483 |
| 170 | 7·25742 | 306·860 7820 | 220 | 2·28386 | 421·358 6695 | 270 | 6·66211 | 540·823 6121 |
| 171 | 1·24102 | 309·093 7781 | 221 | 5·04733 | 423·703 0618 | 271 | 1·80543 | 543·256 5814 |
| 172 | 2·13455 | 311·329 3066 | 222 | 1·12051 | 426·049 4148 | 272 | 4·91078 | 545·691 1503 |
| 173 | 3·69277 | 313·567 3527 | 223 | 2·49873 | 428·397 7197 | 273 | 1·34064 | 548·127 3129 |
| 174 | 6·42543 | 315·807 9019 | 224 | 5·59716 | 430·747 9677 | 274 | 3·67336 | 550·565 0635 |
| 175 | 1·12445 | 318·050 9400 | 225 | 1·25936 | 433·100 1502 | 275 | 1·01017 | 553·004 3962 |
| 176 | 1·97903 | 320·296 4526 | 226 | 2·84616 | 435·454 2586 | 276 | 2·78808 | 555·445 3052 |
| 177 | 3·50289 | 322·544 4259 | 227 | 6·46077 | 437·810 2845 | 277 | 7·72298 | 557·887 7850 |
| 178 | 6·23514 | 324·794 8459 | 228 | 1·47306 | 440·168 2193 | 278 | 2·14699 | 560·331 8298 |
| 179 | 1·11609 | 327·047 6989 | 229 | 3·37330 | 442·528 0548 | 279 | 5·99010 | 562·777 4340 |
| 180 | 2·00896 | 329·302 9714 | 230 | 7·75859 | 444·889 7827 | 280 | 1·67723 | 565·224 5920 |
| 181 | 3·63622 | 331·560 6500 | 231 | 1·79223 | 447·253 3946 | 281 | 4·71301 | 567·673 2984 |
| 182 | 6·61792 | 333·820 7214 | 232 | 4·15798 | 449·618 8826 | 282 | 1·32907 | 570·123 5475 |
| 183 | 1·21108 | 336·083 1725 | 233 | 9·68810 | 451·986 2385 | 283 | 3·76126 | 572·575 3339 |
| 184 | 2·22839 | 338·347 9903 | 234 | 2·26702 | 454·355 4544 | 284 | 1·06820 | 575·028 6523 |
| 185 | 4·12251 | 340·615 1620 | 235 | 5·32749 | 456·726 5223 | 285 | 3·04437 | 577·483 4971 |
| 186 | 7·66787 | 342·884 6750 | 236 | 1·25729 | 459·099 4343 | 286 | 8·70689 | 579·939 8631 |
| 187 | 1·43389 | 345·156 5166 | 237 | 2·97977 | 461·474 1826 | 287 | 2·49888 | 582·397 7450 |
| 188 | 2·69572 | 347·430 6744 | 238 | 7·09185 | 463·850 7596 | 288 | 7·19677 | 584·857 1375 |
| 189 | 5·09491 | 349·707 1362 | 239 | 1·69495 | 466·229 1575 | 289 | 2·07987 | 587·318 0354 |
| 190 | 9·68032 | 351·985 8898 | 240 | 4·06789 | 468·609 3687 | 290 | 6·03161 | 589·780 4334 |
| 191 | 1·84894 | 354·266 9232 | 241 | 9·80360 | 470·991 3857 | 291 | 1·75520 | 592·244 3264 |
| 192 | 3·54997 | 356·550 2244 | 242 | 2·37247 | 473·375 2011 | 292 | 5·12518 | 594·709 7092 |
| 193 | 6·85144 | 358·835 7817 | 243 | 5·76511 | 475·760 8074 | 293 | 1·50168 | 597·176 5768 |
| 194 | 1·32918 | 361·123 5835 | 244 | 1·40669 | 478·148 1972 | 294 | 4·41493 | 599·644 9242 |
| 195 | 2·59190 | 363·413 6181 | 245 | 3·44638 | 480·537 3633 | 295 | 1·30241 | 602·114 7462 |
| 196 | 5·08012 | 365·705 8742 | 246 | 8·47810 | 482·928 2984 | 296 | 3·85512 | 604·586 0379 |
| 197 | 1·00078 | 368·000 3404 | 247 | 2·09409 | 485·320 9954 | 297 | 1·14497 | 607·058 7943 |
| 198 | 1·98155 | 370·297 0056 | 248 | 5·19334 | 487·715 4470 | 298 | 3·41201 | 609·533 0106 |
| 199 | 3·94329 | 372·595 8586 | 249 | 1·29314 | 490·111 6464 | 299 | 1·02019 | 612·008 6818 |
| 200 | 7·88658 | 374·896 8886 | 250 | 3·23286 | 492·509 5864 | 300 | 3·06058 | 614·485 8030 |

For higher values, and non-integral $x > 6$ (or $x \geqslant 1{\cdot}5$ for 4-figure accuracy), use :

$$\log x! = (x+\tfrac{1}{2})\log x + \tfrac{1}{2}\log 2\pi - \left(x - \frac{1}{12x} + \frac{1}{360x^3}\right)\log e.$$

# TABLE XXXI. NATURAL SINES

| | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | 1′ | 2′ | 3′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | ·00000 | 00175 | 00349 | 00524 | 00698 | 00873 | 01047 | 01222 | 01396 | 01571 | 29 | 58 | 87 |
| 1 | ·01745 | 01920 | 02094 | 02269 | 02443 | 02618 | 02792 | 02967 | 03141 | 03316 | 29 | 58 | 87 |
| 2 | ·03490 | 03664 | 03839 | 04013 | 04188 | 04362 | 04536 | 04711 | 04885 | 05059 | 29 | 58 | 87 |
| 3 | ·05234 | 05408 | 05582 | 05756 | 05931 | 06105 | 06279 | 06453 | 06627 | 06802 | 29 | 58 | 87 |
| 4 | ·06976 | 07150 | 07324 | 07498 | 07672 | 07846 | 08020 | 08194 | 08368 | 08542 | 29 | 58 | 87 |
| 5 | ·08716 | 08889 | 09063 | 09237 | 09411 | 09585 | 09758 | 09932 | 10106 | 10279 | 29 | 58 | 87 |
| 6 | ·10453 | 10626 | 10800 | 10973 | 11147 | 11320 | 11494 | 11667 | 11840 | 12014 | 29 | 58 | 87 |
| 7 | ·12187 | 12360 | 12533 | 12706 | 12880 | 13053 | 13226 | 13399 | 13572 | 13744 | 29 | 58 | 86 |
| 8 | ·13917 | 14090 | 14263 | 14436 | 14608 | 14781 | 14954 | 15126 | 15299 | 15471 | 29 | 58 | 86 |
| 9 | ·15643 | 15816 | 15988 | 16160 | 16333 | 16505 | 16677 | 16849 | 17021 | 17193 | 29 | 57 | 86 |
| 10 | ·17365 | 17537 | 17708 | 17880 | 18052 | 18224 | 18395 | 18567 | 18738 | 18910 | 29 | 57 | 86 |
| 11 | ·19081 | 19252 | 19423 | 19595 | 19766 | 19937 | 20108 | 20279 | 20450 | 20620 | 28 | 57 | 86 |
| 12 | ·20791 | 20962 | 21132 | 21303 | 21474 | 21644 | 21814 | 21985 | 22155 | 22325 | 28 | 57 | 85 |
| 13 | ·22495 | 22665 | 22835 | 23005 | 23175 | 23345 | 23514 | 23684 | 23853 | 24023 | 28 | 57 | 85 |
| 14 | ·24192 | 24362 | 24531 | 24700 | 24869 | 25038 | 25207 | 25376 | 25545 | 25713 | 28 | 56 | 85 |
| 15 | ·25882 | 26050 | 26219 | 26387 | 26556 | 26724 | 26892 | 27060 | 27228 | 27396 | 28 | 56 | 84 |
| 16 | ·27564 | 27731 | 27899 | 28067 | 28234 | 28402 | 28569 | 28736 | 28903 | 29070 | 28 | 56 | 84 |
| 17 | ·29237 | 29404 | 29571 | 29737 | 29904 | 30071 | 30237 | 30403 | 30570 | 30736 | 28 | 56 | 83 |
| 18 | ·30902 | 31068 | 31233 | 31399 | 31565 | 31730 | 31896 | 32061 | 32227 | 32392 | 28 | 55 | 83 |
| 19 | ·32557 | 32722 | 32887 | 33051 | 33216 | 33381 | 33545 | 33710 | 33874 | 34038 | 27 | 55 | 82 |
| 20 | ·34202 | 34366 | 34530 | 34694 | 34857 | 35021 | 35184 | 35347 | 35511 | 35674 | 27 | 55 | 82 |
| 21 | ·35837 | 36000 | 36162 | 36325 | 36488 | 36650 | 36812 | 36975 | 37137 | 37299 | 27 | 54 | 81 |
| 22 | ·37461 | 37622 | 37784 | 37946 | 38107 | 38268 | 38430 | 38591 | 38752 | 38912 | 27 | 54 | 81 |
| 23 | ·39073 | 39234 | 39394 | 39555 | 39715 | 39875 | 40035 | 40195 | 40355 | 40514 | 27 | 53 | 80 |
| 24 | ·40674 | 40833 | 40992 | 41151 | 41310 | 41469 | 41628 | 41787 | 41945 | 42104 | 26 | 53 | 79 |
| 25 | ·42262 | 42420 | 42578 | 42736 | 42894 | 43051 | 43209 | 43366 | 43523 | 43680 | 26 | 53 | 79 |
| 26 | ·43837 | 43994 | 44151 | 44307 | 44464 | 44620 | 44776 | 44932 | 45088 | 45243 | 26 | 52 | 78 |
| 27 | ·45399 | 45554 | 45710 | 45865 | 46020 | 46175 | 46330 | 46484 | 46639 | 46793 | 26 | 52 | 77 |
| 28 | ·46947 | 47101 | 47255 | 47409 | 47562 | 47716 | 47869 | 48022 | 48175 | 48328 | 26 | 51 | 77 |
| 29 | ·48481 | 48634 | 48786 | 48938 | 49090 | 49242 | 49394 | 49546 | 49697 | 49849 | 25 | 51 | 76 |
| 30 | ·50000 | 50151 | 50302 | 50453 | 50603 | 50754 | 50904 | 51054 | 51204 | 51354 | 25 | 50 | 75 |
| 31 | ·51504 | 51653 | 51803 | 51952 | 52101 | 52250 | 52399 | 52547 | 52696 | 52844 | 25 | 50 | 74 |
| 32 | ·52992 | 53140 | 53288 | 53435 | 53583 | 53730 | 53877 | 54024 | 54171 | 54317 | 25 | 49 | 74 |
| 33 | ·54464 | 54610 | 54756 | 54902 | 55048 | 55194 | 55339 | 55484 | 55630 | 55775 | 24 | 49 | 73 |
| 34 | ·55919 | 56064 | 56208 | 56353 | 56497 | 56641 | 56784 | 56928 | 57071 | 57215 | 24 | 48 | 72 |
| 35 | ·57358 | 57501 | 57643 | 57786 | 57928 | 58070 | 58212 | 58354 | 58496 | 58637 | 24 | 47 | 71 |
| 36 | ·58779 | 58920 | 59061 | 59201 | 59342 | 59482 | 59622 | 59763 | 59902 | 60042 | 23 | 47 | 70 |
| 37 | ·60182 | 60321 | 60460 | 60599 | 60738 | 60876 | 61015 | 61153 | 61291 | 61429 | 23 | 46 | 69 |
| 38 | ·61566 | 61704 | 61841 | 61978 | 62115 | 62251 | 62388 | 62524 | 62660 | 62796 | 23 | 46 | 68 |
| 39 | ·62932 | 63068 | 63203 | 63338 | 63473 | 63608 | 63742 | 63877 | 64011 | 64145 | 22 | 45 | 67 |
| 40 | ·64279 | 64412 | 64546 | 64679 | 64812 | 64945 | 65077 | 65210 | 65342 | 65474 | 22 | 44 | 66 |
| 41 | ·65606 | 65738 | 65869 | 66000 | 66131 | 66262 | 66393 | 66523 | 66653 | 66783 | 22 | 44 | 65 |
| 42 | ·66913 | 67043 | 67172 | 67301 | 67430 | 67559 | 67688 | 67816 | 67944 | 68072 | 21 | 43 | 64 |
| 43 | ·68200 | 68327 | 68455 | 68582 | 68709 | 68835 | 68962 | 69088 | 69214 | 69340 | 21 | 42 | 63 |
| 44 | ·69466 | 69591 | 69717 | 69842 | 69966 | 70091 | 70215 | 70339 | 70463 | 70587 | 21 | 42 | 62 |

TABLE XXXI. NATURAL SINES—*continued*

|  | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | 1′ | 2′ | 3′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 45 | ·70711 | 70834 | 70957 | 71080 | 71203 | 71325 | 71447 | 71569 | 71691 | 71813 | 20 | 41 | 61 |
| 46 | ·71934 | 72055 | 72176 | 72297 | 72417 | 72537 | 72657 | 72777 | 72897 | 73016 | 20 | 40 | 60 |
| 47 | ·73135 | 73254 | 73373 | 73491 | 73610 | 73728 | 73846 | 73963 | 74080 | 74198 | 20 | 39 | 59 |
| 48 | ·74314 | 74431 | 74548 | 74664 | 74780 | 74896 | 75011 | 75126 | 75241 | 75356 | 19 | 39 | 58 |
| 49 | ·75471 | 75585 | 75700 | 75813 | 75927 | 76041 | 76154 | 76267 | 76380 | 76492 | 19 | 38 | 57 |
| 50 | ·76604 | 76717 | 76828 | 76940 | 77051 | 77162 | 77273 | 77384 | 77494 | 77605 | 19 | 37 | 56 |
| 51 | ·77715 | 77824 | 77934 | 78043 | 78152 | 78261 | 78369 | 78478 | 78586 | 78694 | 18 | 36 | 54 |
| 52 | ·78801 | 78908 | 79016 | 79122 | 79229 | 79335 | 79441 | 79547 | 79653 | 79758 | 18 | 35 | 53 |
| 53 | ·79864 | 79968 | 80073 | 80178 | 80282 | 80386 | 80489 | 80593 | 80696 | 80799 | 17 | 35 | 52 |
| 54 | ·80902 | 81004 | 81106 | 81208 | 81310 | 81412 | 81513 | 81614 | 81714 | 81815 | 17 | 34 | 51 |
| 55 | ·81915 | 82015 | 82115 | 82214 | 82314 | 82413 | 82511 | 82610 | 82708 | 82806 | 16 | 33 | 50 |
| 56 | ·82904 | 83001 | 83098 | 83195 | 83292 | 83389 | 83485 | 83581 | 83676 | 83772 | 16 | 32 | 48 |
| 57 | ·83867 | 83962 | 84057 | 84151 | 84245 | 84339 | 84433 | 84526 | 84619 | 84712 | 16 | 31 | 47 |
| 58 | ·84805 | 84897 | 84989 | 85081 | 85173 | 85264 | 85355 | 85446 | 85536 | 85627 | 15 | 30 | 46 |
| 59 | ·85717 | 85806 | 85896 | 85985 | 86074 | 86163 | 86251 | 86340 | 86427 | 86515 | 15 | 30 | 44 |
| 60 | ·86603 | 86690 | 86777 | 86863 | 86949 | 87036 | 87121 | 87207 | 87292 | 87377 | 14 | 29 | 43 |
| 61 | ·87462 | 87546 | 87631 | 87715 | 87798 | 87882 | 87965 | 88048 | 88130 | 88213 | 14 | 28 | 42 |
| 62 | ·88295 | 88377 | 88458 | 88539 | 88620 | 88701 | 88782 | 88862 | 88942 | 89021 | 13 | 27 | 40 |
| 63 | ·89101 | 89180 | 89259 | 89337 | 89415 | 89493 | 89571 | 89649 | 89726 | 89803 | 13 | 26 | 39 |
| 64 | ·89879 | 89956 | 90032 | 90108 | 90183 | 90259 | 90334 | 90408 | 90483 | 90557 | 13 | 25 | 38 |
| 65 | ·90631 | 90704 | 90778 | 90851 | 90924 | 90996 | 91068 | 91140 | 91212 | 91283 | 12 | 24 | 36 |
| 66 | ·91355 | 91425 | 91496 | 91566 | 91636 | 91706 | 91775 | 91845 | 91914 | 91982 | 12 | 23 | 35 |
| 67 | ·92050 | 92119 | 92186 | 92254 | 92321 | 92388 | 92455 | 92521 | 92587 | 92653 | 11 | 22 | 34 |
| 68 | ·92718 | 92784 | 92849 | 92913 | 92978 | 93042 | 93106 | 93169 | 93232 | 93295 | 11 | 21 | 32 |
| 69 | ·93358 | 93420 | 93483 | 93544 | 93606 | 93667 | 93728 | 93789 | 93849 | 93909 | 10 | 20 | 31 |
| 70 | ·93969 | 94029 | 94088 | 94147 | 94206 | 94264 | 94322 | 94380 | 94438 | 94495 | 10 | 19 | 29 |
| 71 | ·94552 | 94609 | 94665 | 94721 | 94777 | 94832 | 94888 | 94943 | 94997 | 95052 | 9 | 19 | 28 |
| 72 | ·95106 | 95159 | 95213 | 95266 | 95319 | 95372 | 95424 | 95476 | 95528 | 95579 | 9 | 18 | 26 |
| 73 | ·95630 | 95681 | 95732 | 95782 | 95832 | 95882 | 95931 | 95981 | 96029 | 96078 | 8 | 17 | 25 |
| 74 | ·96126 | 96174 | 96222 | 96269 | 96316 | 96363 | 96410 | 96456 | 96502 | 96547 | 8 | 16 | 23 |
| 75 | ·96593 | 96638 | 96682 | 96727 | 96771 | 96815 | 96858 | 96902 | 96945 | 96987 | 7 | 15 | 22 |
| 76 | ·97030 | 97072 | 97113 | 97155 | 97196 | 97237 | 97278 | 97318 | 97358 | 97398 | 7 | 14 | 20 |
| 77 | ·97437 | 97476 | 97515 | 97553 | 97592 | 97630 | 97667 | 97705 | 97742 | 97778 | 6 | 13 | 19 |
| 78 | ·97815 | 97851 | 97887 | 97922 | 97958 | 97992 | 98027 | 98061 | 98096 | 98129 | 6 | 12 | 17 |
| 79 | ·98163 | 98196 | 98229 | 98261 | 98294 | 98325 | 98357 | 98389 | 98420 | 98450 | 5 | 11 | 16 |
| 80 | ·98481 | 98511 | 98541 | 98570 | 98600 | 98629 | 98657 | 98686 | 98714 | 98741 | 5 | 10 | 14 |
| 81 | ·98769 | 98796 | 98823 | 98849 | 98876 | 98902 | 98927 | 98953 | 98978 | 99002 | 4 | 9 | 13 |
| 82 | ·99027 | 99051 | 99075 | 99098 | 99122 | 99144 | 99167 | 99189 | 99211 | 99233 | 4 | 8 | 11 |
| 83 | ·99255 | 99276 | 99297 | 99317 | 99337 | 99357 | 99377 | 99396 | 99415 | 99434 | 3 | 7 | 10 |
| 84 | ·99452 | 99470 | 99488 | 99506 | 99523 | 99540 | 99556 | 99572 | 99588 | 99604 | 3 | 6 | 8 |
| 85 | ·99619 | 99635 | 99649 | 99664 | 99678 | 99692 | 99705 | 99719 | 99731 | 99744 | 2 | 5 | 7 |
| 86 | ·99756 | 99768 | 99780 | 99792 | 99803 | 99813 | 99824 | 99834 | 99844 | 99854 | 2 | 4 | 5 |
| 87 | ·99863 | 99872 | 99881 | 99889 | 99897 | 99905 | 99912 | 99919 | 99926 | 99933 | 1 | 3 | 4 |
| 88 | ·99939 | 99945 | 99951 | 99956 | 99961 | 99966 | 99970 | 99974 | 99978 | 99982 | 1 | 2 | 2 |
| 89 | ·99985 | 99988 | 99990 | 99993 | 99995 | 99996 | 99998 | 99999 | 99999 | 100000 | 0 | 1 | 1 |

## TABLE XXXII. NATURAL TANGENTS

| | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | 1′ | 2′ | 3′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0·00000 | ·00175 | ·00349 | ·00524 | ·00698 | ·00873 | ·01047 | ·01222 | ·01396 | ·01571 | 29 | 58 | 87 |
| 1 | 0·01746 | ·01920 | ·02095 | ·02269 | ·02444 | ·02619 | ·02793 | ·02968 | ·03143 | ·03317 | 29 | 58 | 87 |
| 2 | 0·03492 | ·03667 | ·03842 | ·04016 | ·04191 | ·04366 | ·04541 | ·04716 | ·04891 | ·05066 | 29 | 58 | 87 |
| 3 | 0·05241 | ·05416 | ·05591 | ·05766 | ·05941 | ·06116 | ·06291 | ·06467 | ·06642 | ·06817 | 29 | 58 | 88 |
| 4 | 0·06993 | ·07168 | ·07344 | ·07519 | ·07695 | ·07870 | ·08046 | ·08221 | ·08397 | ·08573 | 29 | 59 | 88 |
| 5 | 0·08749 | ·08925 | ·09101 | ·09277 | ·09453 | ·09629 | ·09805 | ·09981 | ·10158 | ·10334 | 29 | 59 | 88 |
| 6 | 0·10510 | ·10687 | ·10863 | ·11040 | ·11217 | ·11394 | ·11570 | ·11747 | ·11924 | ·12181 | 29 | 59 | 88 |
| 7 | 0·12278 | ·12456 | ·12633 | ·12810 | ·12988 | ·13165 | ·13343 | ·13521 | ·13698 | ·13876 | 30 | 59 | 89 |
| 8 | 0·14054 | ·14232 | ·14410 | ·14588 | ·14767 | ·14945 | ·15124 | ·15302 | ·15481 | ·15660 | 30 | 59 | 89 |
| 9 | 0·15838 | ·16017 | ·16196 | ·16376 | ·16555 | ·16734 | ·16914 | ·17093 | ·17273 | ·17453 | 30 | 60 | 90 |
| 10 | 0·17633 | ·17813 | ·17993 | ·18173 | ·18353 | ·18534 | ·18714 | ·18895 | ·19076 | ·19257 | 30 | 60 | 90 |
| 11 | 0·19438 | ·19619 | ·19801 | ·19982 | ·20164 | ·20345 | ·20527 | ·20709 | ·20891 | ·21073 | 30 | 61 | 91 |
| 12 | 0·21256 | ·21438 | ·21621 | ·21804 | ·21986 | ·22169 | ·22353 | ·22536 | ·22719 | ·22903 | 30 | 61 | 92 |
| 13 | 0·23087 | ·23271 | ·23455 | ·23639 | ·23823 | ·24008 | ·24193 | ·24377 | ·24562 | ·24747 | 31 | 61 | 92 |
| 14 | 0·24933 | ·25118 | ·25304 | ·25490 | ·25676 | ·25862 | ·26048 | ·26235 | ·26421 | ·26608 | 31 | 62 | 93 |
| 15 | 0·26795 | ·26982 | ·27169 | ·27357 | ·27545 | ·27732 | ·27921 | ·28109 | ·28297 | ·28486 | 31 | 63 | 94 |
| 16 | 0·28675 | ·28864 | ·29053 | ·29242 | ·29432 | ·29621 | ·29811 | ·30001 | ·30192 | ·30382 | 32 | 63 | 95 |
| 17 | 0·30573 | ·30764 | ·30955 | ·31147 | ·31338 | ·31530 | ·31722 | ·31914 | ·32106 | ·32299 | 32 | 64 | 96 |
| 18 | 0·32492 | ·32685 | ·32878 | ·33072 | ·33266 | ·33460 | ·33654 | ·33848 | ·34043 | ·34238 | 32 | 65 | 97 |
| 19 | 0·34433 | ·34628 | ·34824 | ·35020 | ·35216 | ·35412 | ·35608 | ·35805 | ·36002 | ·36199 | 33 | 65 | 98 |
| 20 | 0·36397 | ·36595 | ·36793 | ·36991 | ·37190 | ·37388 | ·37588 | ·37787 | ·37986 | ·38186 | 33 | 66 | 99 |
| 21 | 0·38386 | ·38587 | ·38787 | ·38988 | ·39190 | ·39391 | ·39593 | ·39795 | ·39997 | ·40200 | 34 | 67 | 101 |
| 22 | 0·40403 | ·40606 | ·40809 | ·41013 | ·41217 | ·41421 | ·41626 | ·41831 | ·42036 | ·42242 | 34 | 68 | 102 |
| 23 | 0·42447 | ·42654 | ·42860 | ·43067 | ·43274 | ·43481 | ·43689 | ·43897 | ·44105 | ·44314 | 35 | 69 | 104 |
| 24 | 0·44523 | ·44732 | ·44942 | ·45152 | ·45362 | ·45573 | ·45784 | ·45995 | ·46206 | ·46418 | 35 | 70 | 105 |
| 25 | 0·46631 | ·46843 | ·47056 | ·47270 | ·47483 | ·47698 | ·47912 | ·48127 | ·48342 | ·48557 | 36 | 71 | 107 |
| 26 | 0·48773 | ·48989 | ·49206 | ·49423 | ·49640 | ·49858 | ·50076 | ·50295 | ·50514 | ·50733 | 36 | 73 | 109 |
| 27 | 0·50953 | ·51173 | ·51393 | ·51614 | ·51835 | ·52057 | ·52279 | ·52501 | ·52724 | ·52947 | 37 | 74 | 111 |
| 28 | 0·53171 | ·53395 | ·53620 | ·53844 | ·54070 | ·54296 | ·54522 | ·54748 | ·54975 | ·55203 | 38 | 75 | 113 |
| 29 | 0·55431 | ·55659 | ·55888 | ·56117 | ·56347 | ·56577 | ·56808 | ·57039 | ·57271 | ·57503 | 38 | 77 | 115 |
| 30 | 0·57735 | ·57968 | ·58201 | ·58435 | ·58670 | ·58905 | ·59140 | ·59376 | ·59612 | ·59849 | 39 | 78 | 117 |
| 31 | 0·60086 | ·60324 | ·60562 | ·60801 | ·61040 | ·61280 | ·61520 | ·61761 | ·62003 | ·62245 | 40 | 80 | 120 |
| 32 | 0·62487 | ·62730 | ·62973 | ·63217 | ·63462 | ·63707 | ·63953 | ·64199 | ·64446 | ·64693 | 41 | 82 | 123 |
| 33 | 0·64941 | ·65189 | ·65438 | ·65688 | ·65938 | ·66189 | ·66440 | ·66692 | ·66944 | ·67197 | 42 | 84 | 125 |
| 34 | 0·67451 | ·67705 | ·67960 | ·68215 | ·68471 | ·68728 | ·68985 | ·69243 | ·69502 | ·69761 | 43 | 86 | 128 |
| 35 | 0·70021 | ·70281 | ·70542 | ·70804 | ·71066 | ·71329 | ·71593 | ·71857 | ·72122 | ·72388 | 44 | 88 | 131 |
| 36 | 0·72654 | ·72921 | ·73189 | ·73457 | ·73726 | ·73996 | ·74267 | ·74538 | ·74810 | ·75082 | 45 | 90 | 135 |
| 37 | 0·75355 | ·75629 | ·75904 | ·76180 | ·76456 | ·76733 | ·77010 | ·77289 | ·77568 | ·77848 | 46 | 92 | 139 |
| 38 | 0·78129 | ·78410 | ·78692 | ·78975 | ·79259 | ·79544 | ·79829 | ·80115 | ·80402 | ·80690 | 47 | 95 | 142 |
| 39 | 0·80978 | ·81268 | ·81558 | ·81849 | ·82141 | ·82434 | ·82727 | ·83022 | ·83317 | ·83613 | 49 | 98 | 146 |
| 40 | 0·83910 | ·84208 | ·84507 | ·84806 | ·85107 | ·85408 | ·85710 | ·86014 | ·86318 | ·86623 | 50 | 100 | 151 |
| 41 | 0·86929 | ·87236 | ·87543 | ·87852 | ·88162 | ·88473 | ·88784 | ·89097 | ·89410 | ·89725 | 52 | 104 | 155 |
| 42 | 0·90040 | ·90357 | ·90674 | ·90993 | ·91313 | ·91633 | ·91955 | ·92277 | ·92601 | ·92926 | 53 | 107 | 160 |
| 43 | 0·93252 | ·93578 | ·93906 | ·94235 | ·94565 | ·94896 | ·95229 | ·95562 | ·95897 | ·96232 | 55 | 110 | 166 |
| 44 | 0·96569 | ·96907 | ·97246 | ·97586 | ·97927 | ·98270 | ·98613 | ·98958 | ·99304 | ·99652 | 57 | 114 | 171 |

# TABLE XXXII. NATURAL TANGENTS—*continued*

| | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | 1′ | 2′ | 3′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 45 | 1·00000 | ·00350 | ·00701 | ·01053 | ·01406 | ·01761 | ·02117 | ·02474 | ·02832 | ·03192 | 59 | 118 | 177 |
| 46 | 1·03553 | ·03915 | ·04279 | ·04644 | ·05010 | ·05378 | ·05747 | ·06117 | ·06489 | ·06862 | 61 | 123 | 184 |
| 47 | 1·07237 | ·07613 | ·07990 | ·08369 | ·08749 | ·09131 | ·09514 | ·09899 | ·10285 | ·10672 | 64 | 127 | 191 |
| 48 | 1·11061 | ·11452 | ·11844 | ·12238 | ·12633 | ·13029 | ·13428 | ·13828 | ·14229 | ·14632 | 66 | 132 | 198 |
| 49 | 1·15037 | ·15443 | ·15851 | ·16261 | ·16672 | ·17085 | ·17500 | ·17916 | ·18334 | ·18754 | 69 | 138 | 206 |
| 50 | 1·19175 | ·19599 | ·20024 | ·20451 | ·20879 | ·21310 | ·21742 | ·22176 | ·22612 | ·23050 | 72 | 144 | 215 |
| 51 | 1·23490 | ·23931 | ·24375 | ·24820 | ·25268 | ·25717 | ·26169 | ·26622 | ·27077 | ·27535 | 75 | 150 | 225 |
| 52 | 1·27994 | ·28456 | ·28919 | ·29385 | ·29853 | ·30323 | ·30795 | ·31269 | ·31745 | ·32224 | 78 | 157 | 235 |
| 53 | 1·32704 | ·33187 | ·33673 | ·34160 | ·34650 | ·35142 | ·35637 | ·36134 | ·36633 | ·37134 | 82 | 164 | 246 |
| 54 | 1·37638 | ·38145 | ·38653 | ·39165 | ·39679 | ·40195 | ·40714 | ·41235 | ·41759 | ·42286 | 86 | 172 | 258 |
| 55 | 1·42815 | ·43347 | ·43881 | ·44418 | ·44958 | ·45501 | ·46046 | ·46595 | ·47146 | ·47699 | 90 | 181 | 271 |
| 56 | 1·48256 | ·48816 | ·49378 | ·49944 | ·50512 | ·51084 | ·51658 | ·52235 | ·52816 | ·53400 | 95 | 191 | 286 |
| 57 | 1·53986 | ·54576 | ·55170 | ·55766 | ·56366 | ·56969 | ·57575 | ·58184 | ·58797 | ·59414 | 100 | 201 | 302 |
| 58 | 1·60033 | ·60657 | 61283 | ·61914 | ·62548 | ·63185 | ·63826 | ·64471 | ·65120 | ·65772 | 106 | 213 | 319 |
| 59 | 1·66428 | ·67088 | ·67752 | ·68419 | ·69091 | ·69766 | ·70446 | ·71129 | ·71817 | ·72509 | 113 | 225 | 338 |
| 60 | 1·73205 | ·73905 | ·74610 | ·75319 | ·76032 | ·76749 | ·77471 | ·78198 | ·78929 | ·79665 | — | — | — |

## (90°—$\theta$) tan $\theta$

| | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | 1′ | 2′ | 3′ | tan $\theta$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 60 | 51·962 | 51·998 | 52·034 | 52·070 | 52·105 | 52·141 | 52·177 | 52·212 | 52·247 | 52·282 | 6 | 12 | 18 | 1·73 |
| 61 | 52·317 | 52·352 | 52·387 | 52·422 | 52·456 | 52·490 | 52·525 | 52·559 | 52·593 | 52·627 | 6 | 11 | 17 | 1·80 |
| 62 | 52·660 | 52·694 | 52·727 | 52·761 | 52·794 | 52·827 | 52·860 | 52·893 | 52·925 | 52·958 | 6 | 11 | 17 | 1·88 |
| 63 | 52·990 | 53·023 | 53·055 | 53·087 | 53·119 | 53·151 | 53·182 | 53·214 | 53·245 | 53·277 | 5 | 11 | 16 | 1·96 |
| 64 | 53·308 | 53·339 | 53·370 | 53·401 | 53·431 | 53·462 | 53·492 | 53·523 | 53·553 | 53·583 | 5 | 10 | 15 | 2·05 |
| 65 | 53·613 | 53·642 | 53·672 | 53·702 | 53·731 | 53·760 | 53·790 | 53·819 | 53·847 | 53·876 | 5 | 10 | 15 | 2·14 |
| 66 | 53·905 | 53·933 | 53·962 | 53·990 | 54·018 | 54·046 | 54·074 | 54·102 | 54·130 | 54·157 | 5 | 9 | 14 | 2·25 |
| 67 | 54·185 | 54·212 | 54·239 | 54·266 | 54·293 | 54·320 | 54·346 | 54·373 | 54·399 | 54·426 | 4 | 9 | 13 | 2·36 |
| 68 | 54·452 | 54·478 | 54·504 | 54·530 | 54·555 | 54·581 | 54·606 | 54·632 | 54·657 | 54·682 | 4 | 9 | 13 | 2·48 |
| 69 | 54·707 | 54·732 | 54·756 | 54·781 | 54·805 | 54·830 | 54·854 | 54·878 | 54·902 | 54·926 | 4 | 8 | 12 | 2·61 |
| 70 | 54·950 | 54·973 | 54·997 | 55·020 | 55·043 | 55·066 | 55·089 | 55·112 | 55·135 | 55·158 | 4 | 8 | 12 | 2·75 |
| 71 | 55·180 | 55·202 | 55·225 | 55·247 | 55·269 | 55·291 | 55·312 | 55·334 | 55·356 | 55·377 | 4 | 7 | 11 | 2·90 |
| 72 | 55·398 | 55·419 | 55·441 | 55·461 | 55·482 | 55·503 | 55·523 | 55·544 | 55·564 | 55·584 | 3 | 7 | 10 | 3·08 |
| 73 | 55·604 | 55·624 | 55·644 | 55·664 | 55·684 | 55·703 | 55·722 | 55·742 | 55·761 | 55·780 | 3 | 7 | 10 | 3·27 |
| 74 | 55·799 | 55·817 | 55·836 | 55·855 | 55·873 | 55·891 | 55·909 | 55·927 | 55·945 | 55·963 | 3 | 6 | 9 | 3·49 |
| 75 | 55·981 | 55·998 | 56·016 | 56·033 | 56·050 | 56·067 | 56·084 | 56·101 | 56·118 | 56·134 | 3 | 6 | 8 | 3·73 |
| 76 | 56·151 | 56·167 | 56·184 | 56·200 | 56·216 | 56·232 | 56·247 | 56·263 | 56·278 | 56·294 | 3 | 5 | 8 | 4·01 |
| 77 | 56·309 | 56·324 | 56·339 | 56·354 | 56·369 | 56·384 | 56·398 | 56·413 | 56·427 | 56·441 | 2 | 5 | 7 | 4·33 |
| 78 | 56·456 | 56·470 | 56·483 | 56·497 | 56·511 | 56·524 | 56·538 | 56·551 | 56·564 | 56·577 | 2 | 4 | 7 | 4·70 |
| 79 | 56·590 | 56·603 | 56·616 | 56·628 | 56·641 | 56·653 | 56·665 | 56·677 | 56·689 | 56·701 | 2 | 4 | 6 | 5·14 |
| 80 | 56·713 | 56·724 | 56·736 | 56·747 | 56·759 | 56·770 | 56·781 | 56·792 | 56·803 | 56·813 | 2 | 4 | 6 | 5·67 |
| 81 | 56·824 | 56·834 | 56·845 | 56·855 | 56·865 | 56·875 | 56·885 | 56·894 | 56·904 | 56·914 | 2 | 3 | 5 | 6·31 |
| 82 | 56·923 | 56·932 | 56·941 | 56·950 | 56·959 | 56·968 | 56·977 | 56·985 | 56·994 | 57·002 | 1 | 3 | 4 | 7·12 |
| 83 | 57·010 | 57·019 | 57·027 | 57·034 | 57·042 | 57·050 | 57·057 | 57·065 | 57·072 | 57·079 | 1 | 3 | 4 | 8·14 |
| 84 | 57·086 | 57·093 | 57·100 | 57·107 | 57·113 | 57·120 | 57·126 | 57·132 | 57·138 | 57·144 | 1 | 2 | 3 | 9·51 |
| 85 | 57·150 | 57·156 | 57·162 | 57·167 | 57·173 | 57·178 | 57·183 | 57·188 | 57·193 | 57·198 | 1 | 2 | 3 | 11·4 |
| 86 | 57·203 | 57·207 | 57·212 | 57·216 | 57·220 | 57·224 | 57·229 | 57·232 | 57·236 | 57·240 | 1 | 1 | 2 | 14·3 |
| 87 | 57·243 | 57·247 | 57·250 | 57·253 | 57·256 | 57·259 | 57·262 | 57·265 | 57·268 | 57·270 | 0 | 1 | 2 | 19·1 |
| 88 | 57·273 | 57·275 | 57·277 | 57·279 | 57·281 | 57·283 | 57·284 | 57·286 | 57·287 | 57·289 | 0 | 1 | 1 | 28·6 |
| 89 | 57·290 | 57·291 | 57·292 | 57·293 | 57·294 | 57·294 | 57·295 | 57·295 | 57·296 | 57·296 | 0 | 0 | 0 | 57·3 |

## TABLE XXXIII. RANDOM NUMBERS (I)

| | | | | |
|---|---|---|---|---|
| 03 47 43 73 86 | 36 96 47 36 61 | 46 98 63 71 62 | 33 26 16 80 45 | 60 11 14 10 95 |
| 97 74 24 67 62 | 42 81 14 57 20 | 42 53 32 37 32 | 27 07 36 07 51 | 24 51 79 89 73 |
| 16 76 62 27 66 | 56 50 26 71 07 | 32 90 79 78 53 | 13 55 38 58 59 | 88 97 54 14 10 |
| 12 56 85 99 26 | 96 96 68 27 31 | 05 03 72 93 15 | 57 12 10 14 21 | 88 26 49 81 76 |
| 55 59 56 35 64 | 38 54 82 46 22 | 31 62 43 09 90 | 06 18 44 32 53 | 23 83 01 30 30 |
| 16 22 77 94 39 | 49 54 43 54 82 | 17 37 93 23 78 | 87 35 20 96 43 | 84 26 34 91 64 |
| 84 42 17 53 31 | 57 24 55 06 88 | 77 04 74 47 67 | 21 76 33 50 25 | 83 92 12 06 76 |
| 63 01 63 78 59 | 16 95 55 67 19 | 98 10 50 71 75 | 12 86 73 58 07 | 44 39 52 38 79 |
| 33 21 12 34 29 | 78 64 56 07 82 | 52 42 07 44 38 | 15 51 00 13 42 | 99 66 02 79 54 |
| 57 60 86 32 44 | 09 47 27 96 54 | 49 17 46 09 62 | 90 52 84 77 27 | 08 02 73 43 28 |
| 18 18 07 92 46 | 44 17 16 58 09 | 79 83 86 19 62 | 06 76 50 03 10 | 55 23 64 05 05 |
| 26 62 38 97 75 | 84 16 07 44 99 | 83 11 46 32 24 | 20 14 85 88 45 | 10 93 72 88 71 |
| 23 42 40 64 74 | 82 97 77 77 81 | 07 45 32 14 08 | 32 98 94 07 72 | 93 85 79 10 75 |
| 52 36 28 19 95 | 50 92 26 11 97 | 00 56 76 31 38 | 80 22 02 53 53 | 86 60 42 04 53 |
| 37 85 94 35 12 | 83 39 50 08 30 | 42 34 07 96 88 | 54 42 06 87 98 | 35 85 29 48 39 |
| 70 29 17 12 13 | 40 33 20 38 26 | 13 89 51 03 74 | 17 76 37 13 04 | 07 74 21 19 30 |
| 56 62 18 37 35 | 96 83 50 87 75 | 97 12 25 93 47 | 70 33 24 03 54 | 97 77 46 44 80 |
| 99 49 57 22 77 | 88 42 95 45 72 | 16 64 36 16 00 | 04 43 18 66 79 | 94 77 24 21 90 |
| 16 08 15 04 72 | 33 27 14 34 09 | 45 59 34 68 49 | 12 72 07 34 45 | 99 27 72 95 14 |
| 31 16 93 32 43 | 50 27 89 87 19 | 20 15 37 00 49 | 52 85 66 60 44 | 38 68 88 11 80 |
| 68 34 30 13 70 | 55 74 30 77 40 | 44 22 78 84 26 | 04 33 46 09 52 | 68 07 97 06 57 |
| 74 57 25 65 76 | 59 29 97 68 60 | 71 91 38 67 54 | 13 58 18 24 76 | 15 54 55 95 52 |
| 27 42 37 86 53 | 48 55 90 65 72 | 96 57 69 36 10 | 96 46 92 42 45 | 97 60 49 04 91 |
| 00 39 68 29 61 | 66 37 32 20 30 | 77 84 57 03 29 | 10 45 65 04 26 | 11 04 96 67 24 |
| 29 94 98 94 24 | 68 49 69 10 82 | 53 75 91 93 30 | 34 25 20 57 27 | 40 48 73 51 92 |
| 16 90 82 66 59 | 83 62 64 11 12 | 67 19 00 71 74 | 60 47 21 29 68 | 02 02 37 03 31 |
| 11 27 94 75 06 | 06 09 19 74 66 | 02 94 37 34 02 | 76 70 90 30 86 | 38 45 94 30 38 |
| 35 24 10 16 20 | 33 32 51 26 38 | 79 78 45 04 91 | 16 92 53 56 16 | 02 75 50 95 98 |
| 38 23 16 86 38 | 42 38 97 01 50 | 87 75 66 81 41 | 40 01 74 91 62 | 48 51 84 08 32 |
| 31 96 25 91 47 | 96 44 33 49 13 | 34 86 82 53 91 | 00 52 43 48 85 | 27 55 26 89 62 |
| 66 67 40 67 14 | 64 05 71 95 86 | 11 05 65 09 68 | 76 83 20 37 90 | 57 16 00 11 66 |
| 14 90 84 45 11 | 75 73 88 05 90 | 52 27 41 14 86 | 22 98 12 22 08 | 07 52 74 95 80 |
| 68 05 51 18 00 | 33 96 02 75 19 | 07 60 62 93 55 | 59 33 82 43 90 | 49 37 38 44 59 |
| 20 46 78 73 90 | 97 51 40 14 02 | 04 02 33 31 08 | 39 54 16 49 36 | 47 95 93 13 30 |
| 64 19 58 97 79 | 15 06 15 93 20 | 01 90 10 75 06 | 40 78 78 89 62 | 02 67 74 17 33 |
| 05 26 93 70 60 | 22 35 85 15 13 | 92 03 51 59 77 | 59 56 78 06 83 | 52 91 05 70 74 |
| 07 97 10 88 23 | 09 98 42 99 64 | 61 71 62 99 15 | 06 51 29 16 93 | 58 05 77 09 51 |
| 68 71 86 85 85 | 54 87 66 47 54 | 73 32 08 11 12 | 44 95 92 63 16 | 29 56 24 29 48 |
| 26 99 61 65 53 | 58 37 78 80 70 | 42 10 50 67 42 | 32 17 55 85 74 | 94 44 67 16 94 |
| 14 65 52 68 75 | 87 59 36 22 41 | 26 78 63 06 55 | 13 08 27 01 50 | 15 29 39 39 43 |
| 17 53 77 58 71 | 71 41 61 50 72 | 12 41 94 96 26 | 44 95 27 36 99 | 02 96 74 30 83 |
| 90 26 59 21 19 | 23 52 23 33 12 | 96 93 02 18 39 | 07 02 18 36 07 | 25 99 32 70 23 |
| 41 23 52 55 99 | 31 04 49 69 96 | 10 47 48 45 88 | 13 41 43 89 20 | 97 17 14 49 17 |
| 60 20 50 81 69 | 31 99 73 68 68 | 35 81 33 03 76 | 24 30 12 48 60 | 18 99 10 72 34 |
| 91 25 38 05 90 | 94 58 28 41 36 | 45 37 59 03 09 | 90 35 57 29 12 | 82 62 54 65 60 |
| 34 50 57 74 37 | 98 80 33 00 91 | 09 77 93 19 82 | 74 94 80 04 04 | 45 07 31 66 49 |
| 85 22 04 39 43 | 73 81 53 94 79 | 33 62 46 86 28 | 08 31 54 46 31 | 53 94 13 38 47 |
| 09 79 13 77 48 | 73 82 97 22 21 | 05 03 27 24 83 | 72 89 44 05 60 | 35 80 39 94 88 |
| 88 75 80 18 14 | 22 95 75 42 49 | 39 32 82 22 49 | 02 48 07 70 37 | 16 04 61 67 87 |
| 90 96 23 70 00 | 39 00 03 06 90 | 55 85 78 38 36 | 94 37 30 69 32 | 90 89 00 76 33 |

# TABLE XXXIII. RANDOM NUMBERS (II)

| | | | | |
|---|---|---|---|---|
| 53 74 23 99 67 | 61 32 28 69 84 | 94 62 67 86 24 | 98 33 41 19 95 | 47 53 53 38 09 |
| 63 38 06 86 54 | 99 00 65 26 94 | 02 82 90 23 07 | 79 62 67 80 60 | 75 91 12 81 19 |
| 35 30 58 21 46 | 06 72 17 10 94 | 25 21 31 75 96 | 49 28 24 00 49 | 55 65 79 78 07 |
| 63 43 36 82 69 | 65 51 18 37 88 | 61 38 44 12 45 | 32 92 85 88 65 | 54 34 81 85 35 |
| 98 25 37 55 26 | 01 91 82 81 46 | 74 71 12 94 97 | 24 02 71 37 07 | 03 92 18 66 75 |
| 02 63 21 17 69 | 71 50 80 89 56 | 38 15 70 11 48 | 43 40 45 86 98 | 00 83 26 91 03 |
| 64 55 22 21 82 | 48 22 28 06 00 | 61 54 13 43 91 | 82 78 12 23 29 | 06 66 24 12 27 |
| 85 07 26 13 89 | 01 10 07 82 04 | 59 63 69 36 03 | 69 11 15 83 80 | 13 29 54 19 28 |
| 58 54 16 24 15 | 51 54 44 82 00 | 62 61 65 04 69 | 38 18 65 18 97 | 85 72 13 49 21 |
| 34 85 27 84 87 | 61 48 64 56 26 | 90 18 48 13 26 | 37 70 15 42 57 | 65 65 80 39 07 |
| 03 92 18 27 46 | 57 99 16 96 56 | 30 33 72 85 22 | 84 64 38 56 98 | 99 01 30 98 64 |
| 62 95 30 27 59 | 37 75 41 66 48 | 86 97 80 61 45 | 23 53 04 01 63 | 45 76 08 64 27 |
| 08 45 93 15 22 | 60 21 75 46 91 | 98 77 27 85 42 | 28 88 61 08 84 | 69 62 03 42 73 |
| 07 08 55 18 40 | 45 44 75 13 90 | 24 94 96 61 02 | 57 55 66 83 15 | 73 42 37 11 61 |
| 01 85 89 95 66 | 51 10 19 34 88 | 15 84 97 19 75 | 12 76 39 43 78 | 64 63 91 08 25 |
| 72 84 71 14 35 | 19 11 58 49 26 | 50 11 17 17 76 | 86 31 57 20 18 | 95 60 78 46 75 |
| 88 78 28 16 84 | 13 52 53 94 53 | 75 45 69 30 96 | 73 89 65 70 31 | 99 17 43 48 76 |
| 45 17 75 65 57 | 28 40 19 72 12 | 25 12 74 75 67 | 60 40 60 81 19 | 24 62 01 61 16 |
| 96 76 28 12 54 | 22 01 11 94 25 | 71 96 16 16 88 | 68 64 36 74 45 | 19 59 50 88 92 |
| 43 31 67 72 30 | 24 02 94 08 63 | 38 32 36 66 02 | 69 36 38 25 39 | 48 03 45 15 22 |
| 50 44 66 44 21 | 66 06 58 05 62 | 68 15 54 35 02 | 42 35 48 96 32 | 14 52 41 52 48 |
| 22 66 22 15 86 | 26 63 75 41 99 | 58 42 36 72 24 | 58 37 52 18 51 | 03 37 18 39 11 |
| 96 24 40 14 51 | 23 22 30 88 57 | 95 67 47 29 83 | 94 69 40 06 07 | 18 16 36 78 86 |
| 31 73 91 61 19 | 60 20 72 93 48 | 98 57 07 23 69 | 65 95 39 69 58 | 56 80 30 19 44 |
| 78 60 73 99 84 | 43 89 94 36 45 | 56 69 47 07 41 | 90 22 91 07 12 | 78 35 34 08 72 |
| 84 37 90 61 56 | 70 10 23 98 05 | 85 11 34 76 60 | 76 48 45 34 60 | 01 64 18 39 96 |
| 36 67 10 08 23 | 98 93 35 08 86 | 99 29 76 29 81 | 33 34 91 58 93 | 63 14 52 32 52 |
| 07 28 59 07 48 | 89 64 58 89 75 | 83 85 62 27 89 | 30 14 78 56 27 | 86 63 59 80 02 |
| 10 15 83 87 60 | 79 24 31 66 56 | 21 48 24 06 93 | 91 98 94 05 49 | 01 47 59 38 00 |
| 55 19 68 97 65 | 03 73 52 16 56 | 00 53 55 90 27 | 33 42 29 38 87 | 22 13 88 83 34 |
| 53 81 29 13 39 | 35 01 20 71 34 | 62 33 74 82 14 | 53 73 19 09 03 | 56 54 29 56 93 |
| 51 86 32 68 92 | 33 98 74 66 99 | 40 14 71 94 58 | 45 94 19 38 81 | 14 44 99 81 07 |
| 35 91 70 29 13 | 80 03 54 07 27 | 96 94 78 32 66 | 50 95 52 74 33 | 13 80 55 62 54 |
| 37 71 67 95 13 | 20 02 44 95 94 | 64 85 04 05 72 | 01 32 90 76 14 | 53 89 74 60 41 |
| 93 66 13 83 27 | 92 79 64 64 72 | 28 54 96 53 84 | 48 14 52 98 94 | 56 07 93 89 30 |
| 02 96 08 45 65 | 13 05 00 41 84 | 93 07 54 72 59 | 21 45 57 09 77 | 19 48 56 27 44 |
| 49 83 43 48 35 | 82 88 33 69 96 | 72 36 04 19 76 | 47 45 15 18 60 | 82 11 08 95 97 |
| 84 60 71 62 46 | 40 80 81 30 37 | 34 39 23 05 38 | 25 15 35 71 30 | 88 12 57 21 77 |
| 18 17 30 88 71 | 44 91 14 88 47 | 89 23 30 63 15 | 56 34 20 47 89 | 99 82 93 24 98 |
| 79 69 10 61 78 | 71 32 76 95 62 | 87 00 22 58 40 | 92 54 01 75 25 | 43 11 71 99 31 |
| 75 93 36 57 83 | 56 20 14 82 11 | 74 21 97 90 65 | 96 42 68 63 86 | 74 54 13 26 94 |
| 38 30 92 29 03 | 06 28 81 39 38 | 62 25 06 84 63 | 61 29 08 93 67 | 04 32 92 08 09 |
| 51 29 50 10 34 | 31 57 75 95 80 | 51 97 02 74 77 | 76 15 48 49 44 | 18 55 63 77 09 |
| 21 31 38 86 24 | 37 79 81 53 74 | 73 24 16 10 33 | 52 83 90 94 76 | 70 47 14 54 36 |
| 29 01 23 87 88 | 58 02 39 37 67 | 42 10 14 20 92 | 16 55 23 42 45 | 54 96 09 11 06 |
| 95 33 95 22 00 | 18 74 72 00 18 | 38 79 58 69 32 | 81 76 80 26 92 | 82 80 84 25 39 |
| 90 84 60 79 80 | 24 36 59 87 38 | 82 07 53 89 35 | 96 35 23 79 18 | 05 98 90 07 35 |
| 46 40 62 98 82 | 54 97 20 56 95 | 15 74 80 08 32 | 16 46 70 50 80 | 67 72 16 42 79 |
| 20 31 89 03 43 | 38 46 82 68 72 | 32 14 82 99 70 | 80 60 47 18 97 | 63 49 30 21 30 |
| 71 59 73 05 50 | 08 22 23 71 77 | 91 01 93 20 49 | 82 96 59 26 94 | 66 39 67 98 60 |

# TABLE XXXIII. RANDOM NUMBERS (III)

| | | | | |
|---|---|---|---|---|
| 22 17 68 65 84 | 68 95 23 92 35 | 87 02 22 57 51 | 61 09 43 95 06 | 58 24 82 03 47 |
| 19 36 27 59 46 | 13 79 93 37 55 | 39 77 32 77 09 | 85 52 05 30 62 | 47 83 51 62 74 |
| 16 77 23 02 77 | 09 61 87 25 21 | 28 06 24 25 93 | 16 71 13 59 78 | 23 05 47 47 25 |
| 78 43 76 71 61 | 20 44 90 32 64 | 97 67 63 99 61 | 46 38 03 93 22 | 69 81 21 99 21 |
| 03 28 28 26 08 | 73 37 32 04 05 | 69 30 16 09 05 | 88 69 58 28 99 | 35 07 44 75 47 |
| 93 22 53 64 39 | 07 10 63 76 35 | 87 03 04 79 88 | 08 13 13 85 51 | 55 34 57 72 69 |
| 78 76 58 54 74 | 92 38 70 96 92 | 52 06 79 79 45 | 82 63 18 27 44 | 69 66 92 19 09 |
| 23 68 35 26 00 | 99 53 93 61 28 | 52 70 05 48 34 | 56 65 05 61 86 | 90 92 10 70 80 |
| 15 39 25 70 99 | 93 86 52 77 65 | 15 33 59 05 28 | 22 87 26 07 47 | 86 96 98 29 06 |
| 58 71 96 30 24 | 18 46 23 34 27 | 85 13 99 24 44 | 49 18 09 79 49 | 74 16 32 23 02 |
| 57 35 27 33 72 | 24 53 63 94 09 | 41 10 76 47 91 | 44 04 95 49 66 | 39 60 04 59 81 |
| 48 50 86 54 48 | 22 06 34 72 52 | 82 21 15 65 20 | 33 29 94 71 11 | 15 91 29 12 03 |
| 61 96 48 95 03 | 07 16 39 33 66 | 98 56 10 56 79 | 77 21 30 27 12 | 90 49 22 23 62 |
| 36 93 89 41 26 | 29 70 83 63 51 | 99 74 20 52 36 | 87 09 41 15 09 | 98 60 16 03 03 |
| 18 87 00 42 31 | 57 90 12 02 07 | 23 47 37 17 31 | 54 08 01 88 63 | 39 41 88 92 10 |
| 88 56 53 27 59 | 33 35 72 67 47 | 77 34 55 45 70 | 08 18 27 38 90 | 16 95 86 70 75 |
| 09 72 95 84 29 | 49 41 31 06 70 | 42 38 06 45 18 | 64 84 73 31 65 | 52 53 37 97 15 |
| 12 96 88 17 31 | 65 19 69 02 83 | 60 75 86 90 68 | 24 64 19 35 51 | 56 61 87 39 12 |
| 85 94 57 24 16 | 92 09 84 38 76 | 22 00 27 69 85 | 29 81 94 78 70 | 21 94 47 90 12 |
| 38 64 43 59 98 | 98 77 87 68 07 | 91 51 67 62 44 | 40 98 05 93 78 | 23 32 65 41 18 |
| 53 44 09 42 72 | 00 41 86 79 79 | 68 47 22 00 20 | 35 55 31 51 51 | 00 83 63 22 55 |
| 40 76 66 26 84 | 57 99 99 90 37 | 36 63 32 08 58 | 37 40 13 68 97 | 87 64 81 07 83 |
| 02 17 79 18 05 | 12 59 52 57 02 | 22 07 90 47 03 | 28 14 11 30 79 | 20 69 22 40 98 |
| 95 17 82 06 53 | 31 51 10 96 46 | 92 06 88 07 77 | 56 11 50 81 69 | 40 23 72 51 39 |
| 35 76 22 42 92 | 96 11 83 44 80 | 34 68 35 48 77 | 33 42 40 90 60 | 73 96 53 97 86 |
| 26 29 13 56 41 | 85 47 04 66 08 | 34 72 57 59 13 | 82 43 80 46 15 | 38 26 61 70 04 |
| 77 80 20 75 82 | 72 82 32 99 90 | 63 95 73 76 63 | 89 73 44 99 05 | 48 67 26 43 18 |
| 46 40 66 44 52 | 91 36 74 43 53 | 30 82 13 54 00 | 78 45 63 98 35 | 55 03 36 67 68 |
| 37 56 08 18 09 | 77 53 84 46 47 | 31 91 18 95 58 | 24 16 74 11 53 | 44 10 13 85 57 |
| 61 65 61 68 66 | 37 27 47 39 19 | 84 83 70 07 48 | 53 21 40 06 71 | 95 06 79 88 54 |
| 93 43 69 64 07 | 34 18 04 52 35 | 56 27 09 24 86 | 61 85 53 83 45 | 19 90 70 99 00 |
| 21 96 60 12 99 | 11 20 99 45 18 | 48 13 93 55 34 | 18 37 79 49 90 | 65 97 38 20 46 |
| 95 20 47 97 97 | 27 37 83 28 71 | 00 06 41 41 74 | 45 89 09 39 84 | 51 67 11 52 49 |
| 97 86 21 78 73 | 10 65 81 92 59 | 58 76 17 14 97 | 04 76 62 16 17 | 17 95 70 45 80 |
| 69 92 06 34 13 | 59 71 74 17 32 | 27 55 10 24 19 | 23 71 82 13 74 | 63 52 52 01 41 |
| 04 31 17 21 56 | 33 73 99 19 87 | 26 72 39 27 67 | 53 77 57 68 93 | 60 61 97 22 61 |
| 61 06 98 03 91 | 87 14 77 43 96 | 43 00 65 98 50 | 45 60 33 01 07 | 98 99 46 50 47 |
| 85 93 85 86 88 | 72 87 08 62 40 | 16 06 10 89 20 | 23 21 34 74 97 | 76 38 03 29 63 |
| 21 74 32 47 45 | 73 96 07 94 52 | 09 65 90 77 47 | 25 76 16 19 33 | 53 05 70 53 30 |
| 15 69 53 82 80 | 79 96 23 53 10 | 65 39 07 16 29 | 45 33 02 43 70 | 02 87 40 41 45 |
| 02 89 08 04 49 | 20 21 14 68 86 | 87 63 93 95 17 | 11 29 01 95 80 | 35 14 97 35 33 |
| 87 18 15 89 79 | 85 43 01 72 73 | 08 61 74 51 69 | 89 74 39 82 15 | 94 51 33 41 67 |
| 98 83 71 94 22 | 59 97 50 99 52 | 08 52 85 08 40 | 87 80 61 65 31 | 91 51 80 32 44 |
| 10 08 58 21 66 | 72 68 49 29 31 | 89 85 84 46 06 | 59 73 19 85 23 | 65 09 29 75 63 |
| 47 90 56 10 08 | 88 02 84 27 83 | 42 29 72 23 19 | 66 56 45 65 79 | 20 71 53 20 25 |
| 22 85 61 68 90 | 49 64 92 85 44 | 16 40 12 89 88 | 50 14 49 81 06 | 01 82 77 45 12 |
| 67 80 43 79 33 | 12 83 11 41 16 | 25 58 19 68 70 | 77 02 54 00 52 | 53 43 37 15 26 |
| 27 62 50 96 72 | 79 44 61 40 15 | 14 53 40 65 39 | 27 31 58 50 28 | 11 39 03 34 25 |
| 33 78 80 87 15 | 38 30 06 38 21 | 14 47 47 07 26 | 54 96 87 53 32 | 40 36 40 96 76 |
| 13 13 92 66 99 | 47 24 49 57 74 | 32 25 43 62 17 | 10 97 11 69 84 | 99 63 22 32 98 |

# TABLE XXXIII. RANDOM NUMBERS (IV)

| | | | | |
|---|---|---|---|---|
| 10 27 53 96 23 | 71 50 54 36 23 | 54 31 04 82 98 | 04 14 12 15 09 | 26 78 25 47 47 |
| 28 41 50 61 88 | 64 85 27 20 18 | 83 36 36 05 56 | 39 71 65 09 62 | 94 76 62 11 89 |
| 34 21 42 57 02 | 59 19 18 97 48 | 80 30 03 30 98 | 05 24 67 70 07 | 84 97 50 87 46 |
| 61 81 77 23 23 | 82 82 11 54 08 | 53 28 70 58 96 | 44 07 39 55 43 | 42 34 43 39 28 |
| 61 15 18 13 54 | 16 86 20 26 88 | 90 74 80 55 09 | 14 53 90 51 17 | 52 01 63 01 59 |
| 91 76 21 64 64 | 44 91 13 32 97 | 75 31 62 66 54 | 84 80 32 75 77 | 56 08 25 70 29 |
| 00 97 79 08 06 | 37 30 28 59 85 | 53 56 68 53 40 | 01 74 39 59 73 | 30 19 99 85 48 |
| 36 46 18 34 94 | 75 20 80 27 77 | 78 91 69 16 00 | 08 43 18 73 68 | 67 69 61 34 25 |
| 88 98 99 60 50 | 65 95 79 42 94 | 93 62 40 89 96 | 43 56 47 71 66 | 46 76 29 67 02 |
| 04 37 59 87 21 | 05 02 03 24 17 | 47 97 81 56 51 | 92 34 86 01 82 | 55 51 33 12 91 |
| 63 62 06 34 41 | 94 21 78 55 09 | 72 76 45 16 94 | 29 95 81 83 83 | 79 88 01 97 30 |
| 78 47 23 53 90 | 34 41 92 45 71 | 09 23 70 70 07 | 12 38 92 79 43 | 14 85 11 47 23 |
| 87 68 62 15 43 | 53 14 36 59 25 | 54 47 33 70 15 | 59 24 48 40 35 | 50 03 42 99 36 |
| 47 60 92 10 77 | 88 59 53 11 52 | 66 25 69 07 04 | 48 68 64 71 06 | 61 65 70 22 12 |
| 56 88 87 59 41 | 65 28 04 67 53 | 95 79 88 37 31 | 50 41 06 94 76 | 81 83 17 16 33 |
| 02 57 45 86 67 | 73 43 07 34 48 | 44 26 87 93 29 | 77 09 61 67 84 | 06 69 44 77 75 |
| 31 54 14 13 17 | 48 62 11 90 60 | 68 12 93 64 28 | 46 24 79 16 76 | 14 60 25 51 01 |
| 28 50 16 43 36 | 28 97 85 58 99 | 67 22 52 76 23 | 24 70 36 54 54 | 59 28 61 71 96 |
| 63 29 62 66 50 | 02 63 45 52 38 | 67 63 47 54 75 | 83 24 78 43 20 | 92 63 13 47 48 |
| 45 65 58 26 51 | 76 96 59 38 72 | 86 57 45 71 46 | 44 67 76 14 55 | 44 88 01 62 12 |
| 39 65 36 63 70 | 77 45 85 50 51 | 74 13 39 35 22 | 30 53 36 02 95 | 49 34 88 73 61 |
| 73 71 98 16 04 | 29 18 94 51 23 | 76 51 94 84 86 | 79 93 96 38 63 | 08 58 25 58 94 |
| 72 20 56 20 11 | 72 65 71 08 86 | 79 57 95 13 91 | 97 48 72 66 48 | 09 71 17 24 89 |
| 75 17 26 99 76 | 89 37 20 70 01 | 77 31 61 95 46 | 26 97 05 73 51 | 53 33 18 72 87 |
| 37 48 60 82 29 | 81 30 15 39 14 | 48 38 75 93 29 | 06 87 37 78 48 | 45 56 00 84 47 |
| 68 08 02 80 72 | 83 71 46 30 49 | 89 17 95 88 29 | 02 39 56 03 46 | 97 74 06 56 17 |
| 14 23 98 61 67 | 70 52 85 01 50 | 01 84 02 78 43 | 10 62 98 19 41 | 18 83 99 47 99 |
| 49 08 96 21 44 | 25 27 99 41 28 | 07 41 08 34 66 | 19 42 74 39 91 | 41 96 53 78 72 |
| 78 37 06 08 43 | 63 61 62 42 29 | 39 68 95 10 96 | 09 24 23 00 62 | 56 12 80 73 16 |
| 37 21 34 17 68 | 68 96 83 23 56 | 32 84 60 15 31 | 44 73 67 34 77 | 91 15 79 74 58 |
| 14 29 09 34 04 | 87 83 07 55 07 | 76 58 30 83 64 | 87 29 25 58 84 | 86 50 60 00 25 |
| 58 43 28 06 36 | 49 52 83 51 14 | 47 56 91 29 34 | 05 87 31 06 95 | 12 45 57 09 09 |
| 10 43 67 29 70 | 80 62 80 03 42 | 10 80 21 38 84 | 90 56 35 03 09 | 43 12 74 49 14 |
| 44 38 88 39 54 | 86 97 37 44 22 | 00 95 01 31 76 | 17 16 29 56 63 | 38 78 94 49 81 |
| 90 69 59 19 51 | 85 39 52 85 13 | 07 28 37 07 61 | 11 16 36 27 03 | 78 86 72 04 95 |
| 41 47 10 25 62 | 97 05 31 03 61 | 20 26 36 31 62 | 68 69 86 95 44 | 84 95 48 46 45 |
| 91 94 14 63 19 | 75 89 11 47 11 | 31 56 34 19 09 | 79 57 92 36 59 | 14 93 87 81 40 |
| 80 06 54 18 66 | 09 18 94 06 19 | 98 40 07 17 81 | 22 45 44 84 11 | 24 62 20 42 31 |
| 67 72 77 63 48 | 84 08 31 55 58 | 24 33 45 77 58 | 80 45 67 93 82 | 75 70 16 08 24 |
| 59 40 24 13 27 | 79 26 88 86 30 | 01 31 60 10 39 | 53 58 47 70 93 | 85 81 56 39 38 |
| 05 90 35 89 95 | 01 61 16 96 94 | 50 78 13 69 36 | 37 68 53 37 31 | 71 26 35 03 71 |
| 44 43 80 69 98 | 46 68 05 14 82 | 90 78 50 05 62 | 77 79 13 57 44 | 59 60 10 39 66 |
| 61 81 31 96 82 | 00 57 25 60 59 | 46 72 60 18 77 | 55 66 12 62 11 | 08 99 55 64 57 |
| 42 88 07 10 05 | 24 98 65 63 21 | 47 21 61 88 32 | 27 80 30 21 60 | 10 92 35 36 12 |
| 77 94 30 05 39 | 28 10 99 00 27 | 12 73 73 99 12 | 49 99 57 94 82 | 96 88 57 17 91 |
| 78 83 19 76 16 | 94 11 68 84 26 | 23 54 20 86 85 | 23 86 66 99 07 | 36 37 34 92 09 |
| 87 76 59 61 81 | 43 63 64 61 61 | 65 76 36 95 90 | 18 48 27 45 68 | 27 23 65 30 72 |
| 91 43 05 96 47 | 55 78 99 95 24 | 37 55 85 78 78 | 01 48 41 19 10 | 35 19 54 07 73 |
| 84 97 77 72 73 | 09 62 06 65 72 | 87 12 49 03 60 | 41 15 20 76 27 | 50 47 02 29 16 |
| 87 41 60 76 83 | 44 88 96 07 80 | 83 05 83 38 96 | 73 70 66 81 90 | 30 56 10 48 59 |

## Table XXXIII. Random Numbers (V)

| | | | | |
|---|---|---|---|---|
| 28 89 65 87 08 | 13 50 63 04 23 | 25 47 57 91 13 | 52 62 24 19 94 | 91 67 48 57 10 |
| 30 29 43 65 42 | 78 66 28 55 80 | 47 46 41 90 08 | 55 98 78 10 70 | 49 92 05 12 07 |
| 95 74 62 60 53 | 51 57 32 22 27 | 12 72 72 27 77 | 44 67 32 23 13 | 67 95 07 76 30 |
| 01 85 54 96 72 | 66 86 65 64 60 | 56 59 75 36 75 | 46 44 33 63 71 | 54 50 06 44 75 |
| 10 91 46 96 86 | 19 83 52 47 53 | 65 00 51 93 51 | 30 80 05 19 29 | 56 23 27 19 03 |
| 05 33 18 08 51 | 51 78 57 26 17 | 34 87 96 23 95 | 89 99 93 39 79 | 11 28 94 15 52 |
| 04 43 13 37 00 | 79 68 96 26 60 | 70 39 83 66 56 | 62 03 55 86 57 | 77 55 33 62 02 |
| 05 85 40 25 24 | 73 52 93 70 50 | 48 21 47 74 63 | 17 27 27 51 26 | 35 96 29 00 45 |
| 84 90 90 65 77 | 63 99 25 69 02 | 09 04 03 35 78 | 19 79 95 07 21 | 02 84 48 51 97 |
| 28 55 53 09 48 | 86 28 30 02 35 | 71 30 32 06 47 | 93 74 21 86 33 | 49 90 21 69 74 |
| 89 83 40 69 80 | 97 96 47 59 97 | 56 33 24 87 36 | 17 18 16 90 46 | 75 27 28 52 13 |
| 73 20 96 05 68 | 93 41 69 96 07 | 97 50 81 79 59 | 42 37 13 81 83 | 92 42 85 04 31 |
| 10 89 07 76 21 | 40 24 74 36 42 | 40 33 04 46 24 | 35 63 02 31 61 | 34 59 43 36 96 |
| 91 50 27 78 37 | 06 06 16 25 98 | 17 78 80 36 85 | 26 41 77 63 37 | 71 63 94 94 33 |
| 03 45 44 66 88 | 97 81 26 03 89 | 39 46 67 21 17 | 98 10 39 33 15 | 61 63 00 25 92 |
| 89 41 58 91 63 | 65 99 59 97 84 | 90 14 79 61 55 | 56 16 88 87 60 | 32 15 99 67 43 |
| 13 43 00 97 26 | 16 91 21 32 41 | 60 22 66 72 17 | 31 85 33 69 07 | 68 49 20 43 29 |
| 71 71 00 51 72 | 62 03 89 26 32 | 35 27 99 18 25 | 78 12 03 09 70 | 50 93 19 35 56 |
| 19 28 15 00 41 | 92 27 73 40 38 | 37 11 05 75 16 | 98 81 99 37 29 | 92 20 32 39 67 |
| 56 38 30 92 30 | 45 51 94 69 04 | 00 84 14 36 37 | 95 66 39 01 09 | 21 68 40 95 79 |
| 39 27 52 89 11 | 00 81 06 28 48 | 12 08 05 75 26 | 03 35 63 05 77 | 13 81 20 67 58 |
| 73 13 28 58 01 | 05 06 42 24 07 | 60 60 29 99 93 | 72 93 78 04 36 | 25 76 01 54 03 |
| 81 60 84 51 57 | 12 68 46 55 89 | 60 09 71 87 89 | 70 81 10 95 91 | 83 79 68 20 66 |
| 05 62 98 07 85 | 07 79 26 69 61 | 67 85 72 37 41 | 85 79 76 48 23 | 61 58 87 08 05 |
| 62 97 16 29 18 | 52 16 16 23 56 | 62 95 80 97 63 | 32 25 34 03 36 | 48 84 60 37 65 |
| 31 13 63 21 08 | 16 01 92 58 21 | 48 79 74 73 72 | 08 64 80 91 38 | 07 28 66 61 59 |
| 97 38 35 34 19 | 89 84 05 34 47 | 88 09 31 54 88 | 97 96 86 01 69 | 46 13 95 65 96 |
| 32 11 78 33 82 | 51 99 98 44 39 | 12 75 10 60 36 | 80 66 39 94 97 | 42 36 31 16 59 |
| 81 99 13 37 05 | 08 12 60 39 23 | 61 73 84 89 18 | 26 02 04 37 95 | 96 18 69 06 30 |
| 45 74 00 03 05 | 69 99 47 26 52 | 48 06 30 00 18 | 03 30 28 55 59 | 66 10 71 44 05 |
| 11 84 13 69 01 | 88 91 28 79 50 | 71 42 14 96 55 | 98 59 96 01 36 | 88 77 90 45 59 |
| 14 66 12 87 22 | 59 45 27 08 51 | 85 64 23 85 41 | 64 72 08 59 44 | 67 98 36 65 56 |
| 40 25 67 87 82 | 84 27 17 30 37 | 48 69 49 02 58 | 98 02 50 58 11 | 95 39 06 35 63 |
| 44 48 97 49 43 | 65 45 53 41 07 | 14 83 46 74 11 | 76 66 63 60 08 | 90 54 33 65 84 |
| 41 94 54 06 57 | 48 28 01 83 84 | 09 11 21 91 73 | 97 28 44 74 06 | 22 30 95 69 72 |
| 07 12 15 58 84 | 93 18 31 83 45 | 54 52 62 29 91 | 53 58 54 66 05 | 47 19 63 92 75 |
| 64 27 90 43 52 | 18 26 32 96 83 | 50 58 45 27 57 | 14 96 39 64 85 | 73 87 96 76 23 |
| 80 71 86 41 03 | 45 62 63 40 88 | 35 69 34 10 94 | 32 22 52 04 74 | 69 63 21 83 41 |
| 27 06 08 09 92 | 26 22 59 28 27 | 38 58 22 14 79 | 24 32 12 38 42 | 33 56 90 92 57 |
| 54 68 97 20 54 | 33 26 74 03 30 | 74 22 19 13 48 | 30 28 01 92 49 | 58 61 52 27 03 |
| 02 92 65 68 99 | 05 53 15 26 70 | 04 69 22 64 07 | 04 73 25 74 82 | 78 35 22 21 88 |
| 83 52 57 78 62 | 98 61 70 48 22 | 68 50 64 55 75 | 42 70 32 09 60 | 58 70 61 43 97 |
| 82 82 76 31 33 | 85 13 41 38 10 | 16 47 61 43 77 | 83 27 19 70 41 | 34 78 77 60 25 |
| 38 61 34 09 49 | 04 41 66 09 76 | 20 50 73 40 95 | 24 77 95 73 20 | 47 42 80 61 03 |
| 01 01 11 88 38 | 03 10 16 82 24 | 39 58 20 12 39 | 82 77 02 18 88 | 33 11 49 15 16 |
| 21 66 14 38 28 | 54 08 18 07 04 | 92 17 63 36 75 | 33 14 11 11 78 | 97 30 53 62 38 |
| 32 29 30 69 59 | 68 50 33 31 47 | 15 64 88 75 27 | 04 51 41 61 96 | 86 62 93 66 71 |
| 04 59 21 65 47 | 39 90 89 86 77 | 46 86 86 88 86 | 50 09 13 24 91 | 54 80 67 78 66 |
| 38 64 50 07 36 | 56 50 45 94 25 | 48 28 48 30 51 | 60 73 73 03 87 | 68 47 37 10 84 |
| 48 33 50 83 53 | 59 77 64 59 90 | 58 92 62 50 18 | 93 09 45 89 06 | 13 26 98 86 29 |

# TABLE XXXIII. RANDOM NUMBERS (VI)

| | | | | |
|---|---|---|---|---|
| 25 19 64 82 84 | 62 74 29 92 24 | 61 03 91 22 48 | 64 94 63 15 07 | 66 85 12 00 27 |
| 23 02 41 46 04 | 44 31 52 43 07 | 44 06 03 09 34 | 19 83 94 62 94 | 48 28 01 51 92 |
| 55 85 66 96 28 | 28 30 62 58 83 | 65 68 62 42 45 | 13 08 60 46 28 | 95 68 45 52 43 |
| 68 45 19 69 59 | 35 14 82 56 80 | 22 06 52 26 39 | 59 78 98 76 14 | 36 09 03 01 86 |
| 69 31 46 29 85 | 18 88 26 95 54 | 01 02 14 03 05 | 48 00 26 43 85 | 33 93 81 45 95 |
| 37 31 61 28 98 | 94 61 47 03 10 | 67 80 84 41 26 | 88 84 59 69 14 | 77 32 82 81 89 |
| 66 42 19 24 94 | 13 13 38 69 96 | 76 69 76 24 13 | 43 83 10 13 24 | 18 32 84 85 04 |
| 33 65 78 12 35 | 91 59 11 38 44 | 23 31 48 75 74 | 05 30 08 46 32 | 90 04 93 56 16 |
| 76 32 06 19 35 | 22 95 30 19 29 | 57 74 43 20 90 | 20 25 36 70 69 | 38 32 11 01 01 |
| 43 33 42 02 59 | 20 39 84 95 61 | 58 22 04 02 99 | 99 78 78 83 82 | 43 67 16 38 95 |
| 28 31 93 43 94 | 87 73 19 38 47 | 54 36 90 98 10 | 83 43 32 26 26 | 22 00 90 59 22 |
| 97 19 21 63 34 | 69 33 17 03 02 | 11 15 50 46 08 | 42 69 60 17 42 | 14 68 61 14 48 |
| 82 80 37 14 20 | 56 39 59 89 63 | 33 90 38 44 50 | 78 22 87 10 88 | 06 58 87 39 67 |
| 03 68 03 13 60 | 64 13 09 37 11 | 86 02 57 41 99 | 31 66 60 65 64 | 03 03 02 58 97 |
| 65 16 58 11 01 | 98 78 80 63 23 | 07 37 66 20 56 | 20 96 06 79 80 | 33 39 40 49 42 |
| 24 65 58 57 04 | 18 62 85 28 24 | 26 45 17 82 76 | 39 65 01 73 91 | 50 37 49 38 73 |
| 02 72 64 07 75 | 85 66 48 38 73 | 75 10 96 59 31 | 48 78 58 08 88 | 72 08 54 57 17 |
| 79 16 78 63 99 | 43 61 00 66 42 | 76 26 71 14 33 | 33 86 76 71 66 | 37 85 05 56 07 |
| 04 75 14 93 39 | 68 52 16 83 34 | 64 09 44 62 58 | 48 32 72 26 95 | 32 67 35 49 71 |
| 40 64 64 57 60 | 97 00 12 91 33 | 22 14 73 01 11 | 83 97 68 95 65 | 67 77 80 98 87 |
| 06 27 07 34 26 | 01 52 48 69 57 | 19 17 53 55 96 | 02 41 03 89 33 | 86 85 73 02 32 |
| 62 40 03 87 10 | 96 88 22 46 94 | 35 56 60 94 20 | 60 73 04 84 98 | 96 45 18 47 07 |
| 00 98 48 18 97 | 91 51 63 27 95 | 74 25 84 03 07 | 88 29 04 79 84 | 03 71 13 78 26 |
| 50 64 19 18 91 | 98 55 83 46 09 | 49 66 41 12 45 | 41 49 36 83 43 | 53 75 35 13 39 |
| 38 54 52 25 78 | 01 98 00 89 85 | 86 12 22 89 25 | 10 10 71 19 45 | 88 84 77 00 07 |
| 46 86 80 97 78 | 65 12 64 64 70 | 58 41 05 49 08 | 68 68 88 54 00 | 81 61 61 80 41 |
| 90 72 92 93 10 | 09 12 81 93 63 | 69 30 02 04 26 | 92 36 48 69 45 | 91 99 08 07 65 |
| 66 21 41 77 60 | 99 35 72 61 22 | 52 40 74 67 29 | 97 50 71 39 79 | 57 82 14 88 06 |
| 87 05 46 52 76 | 89 96 34 22 37 | 27 11 57 04 19 | 57 93 08 35 69 | 07 51 19 92 66 |
| 46 90 61 03 06 | 89 85 33 22 80 | 34 89 12 29 37 | 44 71 38 40 37 | 15 49 55 51 08 |
| 11 88 53 06 09 | 81 83 33 98 29 | 91 27 59 43 09 | 70 72 51 49 73 | 35 97 25 83 41 |
| 11 05 92 06 97 | 68 82 34 08 83 | 25 40 58 40 64 | 56 42 78 54 06 | 60 96 96 12 82 |
| 33 94 24 20 28 | 62 42 07 12 63 | 34 39 02 92 31 | 80 61 68 44 19 | 09 92 14 73 49 |
| 24 89 74 75 61 | 61 02 73 36 85 | 67 28 50 49 85 | 37 79 95 02 66 | 73 19 76 28 13 |
| 15 19 74 67 23 | 61 38 93 73 68 | 76 23 15 58 20 | 35 36 82 82 59 | 01 33 48 17 66 |
| 05 64 12 70 88 | 80 58 35 06 88 | 73 48 27 39 43 | 43 40 13 35 45 | 55 10 54 38 50 |
| 57 49 36 44 06 | 74 93 55 39 26 | 27 70 98 76 68 | 78 36 26 24 06 | 43 24 56 40 80 |
| 77 82 96 96 97 | 60 42 17 18 48 | 16 34 92 19 52 | 98 84 48 42 92 | 83 19 06 77 78 |
| 24 10 70 06 51 | 59 62 37 95 42 | 53 67 14 95 29 | 84 65 43 07 30 | 77 54 00 15 42 |
| 50 00 07 78 23 | 49 54 36 85 14 | 18 50 54 18 82 | 23 79 80 71 37 | 60 62 95 40 30 |
| 44 37 76 21 96 | 37 03 08 98 64 | 90 85 59 43 64 | 17 79 96 52 35 | 21 05 22 59 30 |
| 90 57 55 17 47 | 53 26 79 20 38 | 69 90 58 64 03 | 33 48 32 91 54 | 68 44 90 24 25 |
| 50 74 64 67 42 | 95 28 12 73 23 | 32 54 98 64 94 | 82 17 18 17 14 | 55 10 61 64 29 |
| 44 04 70 22 02 | 84 31 64 64 08 | 52 55 04 24 29 | 91 95 43 81 14 | 66 13 18 47 44 |
| 32 74 61 64 73 | 21 46 51 44 77 | 72 48 92 00 05 | 83 59 89 65 06 | 53 76 70 58 78 |
| 75 73 51 70 49 | 12 53 67 51 54 | 38 10 11 67 73 | 22 32 61 43 75 | 31 61 22 21 11 |
| 76 18 36 16 34 | 16 28 25 82 98 | 64 26 70 54 87 | 49 48 55 11 39 | 94 25 20 80 85 |
| 00 17 37 71 81 | 64 21 91 15 82 | 81 04 14 52 11 | 39 07 30 60 77 | 39 18 27 85 68 |
| 54 95 57 55 04 | 12 77 40 70 14 | 79 86 61 57 50 | 52 49 41 73 46 | 05 63 34 92 33 |
| 69 99 95 54 63 | 44 37 33 53 17 | 38 06 58 37 93 | 47 10 62 31 28 | 63 59 40 40 32 |

# TABLE XXXIII1. RANDOM PERMUTATIONS OF 10 NUMBERS

| | 0<br>5 | 1<br>6 | 2<br>7 | 3<br>8 | 4<br>9 | 5<br>0 | 6<br>1 | 7<br>2 | 8<br>3 | 9<br>4 | 0<br>5 | 1<br>6 | 2<br>7 | 3<br>8 | 4<br>9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 03528 | 28071 | 97041 | 45167 | 35421 | 71345 | 47286 | 83567 | 94170 | 46789 | 56471 | 65132 | 18294 | 73618 | 89714 |
| | 49761 | 39465 | 52683 | 82093 | 09867 | 06982 | 10359 | 49102 | 65328 | 02531 | 93208 | 74809 | 50376 | 04952 | 02365 |
| 15 | 80219 | 29485 | 85093 | 45687 | 32579 | 52867 | 70851 | 03621 | 51490 | 53180 | 76283 | 90287 | 34605 | 67295 | 82906 |
| | 43657 | 70361 | 24176 | 30129 | 48016 | 03194 | 29463 | 94587 | 32768 | 76924 | 90451 | 56314 | 12879 | 10834 | 45137 |
| 30 | 36524 | 07561 | 67820 | 20739 | 69045 | 54978 | 90317 | 64529 | 57319 | 74861 | 94816 | 10798 | 06431 | 53984 | 07196 |
| | 97801 | 98234 | 13459 | 58146 | 13782 | 36102 | 48562 | 78301 | 64802 | 39502 | 05723 | 32546 | 29857 | 16072 | 32584 |
| 45 | 56913 | 06289 | 64527 | 30265 | 43209 | 84152 | 69183 | 01278 | 41973 | 03471 | 85014 | 01628 | 42016 | 90164 | 90418 |
| | 84027 | 51734 | 80193 | 49718 | 81675 | 69307 | 47250 | 39546 | 25068 | 56829 | 27639 | 73954 | 58379 | 53827 | 73625 |
| 60 | 70523 | 83297 | 81239 | 05913 | 75206 | 18350 | 81623 | 59327 | 08317 | 39581 | 45736 | 12987 | 40568 | 20915 | 40819 |
| | 68194 | 60541 | 06745 | 68742 | 38491 | 67924 | 07549 | 64081 | 56492 | 60247 | 19802 | 54360 | 23917 | 48367 | 52367 |
| 75 | 59208 | 56078 | 74268 | 54981 | 12573 | 51246 | 43985 | 04628 | 54670 | 93108 | 86341 | 10678 | 70468 | 53920 | 91478 |
| | 13674 | 31492 | 09351 | 27630 | 68490 | 38097 | 27160 | 97531 | 32918 | 67542 | 95207 | 42593 | 51239 | 74618 | 32650 |
| 90 | 40329 | 15237 | 65897 | 18693 | 43709 | 10783 | 41237 | 72168 | 87194 | 80472 | 38125 | 23945 | 73981 | 31749 | 52831 |
| | 71568 | 98604 | 13402 | 20547 | 58612 | 95426 | 89605 | 35904 | 23506 | 69351 | 46970 | 10687 | 06254 | 26580 | 49760 |
| 105 | 78136 | 62531 | 92143 | 28935 | 98410 | 78059 | 76901 | 24618 | 85102 | 30642 | 10378 | 61739 | 41980 | 59761 | 65943 |
| | 09452 | 84097 | 05876 | 46107 | 75236 | 61432 | 35482 | 57093 | 93647 | 15987 | 24956 | 08245 | 25763 | 42803 | 80172 |
| 120 | 58236 | 29871 | 53107 | 50762 | 47103 | 81973 | 35716 | 83924 | 21593 | 98432 | 95260 | 83496 | 27810 | 82193 | 24851 |
| | 10974 | 64350 | 89264 | 41839 | 69852 | 65042 | 42089 | 67501 | 40678 | 57601 | 84371 | 17250 | 95634 | 67450 | 90367 |
| 135 | 83417 | 26179 | 19562 | 71358 | 78542 | 72896 | 74312 | 57346 | 16248 | 68102 | 76581 | 73294 | 02698 | 72850 | 35710 |
| | 29065 | 45038 | 38704 | 90246 | 13069 | 43105 | 85960 | 82019 | 97305 | 53794 | 24309 | 86015 | 43517 | 94316 | 98624 |
| 150 | 48210 | 25789 | 52193 | 36489 | 82167 | 01325 | 52839 | 25301 | 24150 | 71306 | 06417 | 23057 | 80129 | 82439 | 34620 |
| | 79365 | 31460 | 04687 | 17502 | 49053 | 74896 | 70614 | 78649 | 63987 | 94258 | 58329 | 94168 | 65374 | 16075 | 87915 |
| 165 | 67439 | 45961 | 21035 | 64530 | 84067 | 32516 | 57819 | 23461 | 15894 | 20935 | 37198 | 67983 | 94786 | 96408 | 58037 |
| | 10285 | 80723 | 64789 | 81972 | 51923 | 07498 | 04623 | 80957 | 06723 | 78641 | 65240 | 02514 | 02315 | 57123 | 92164 |
| 180 | 25481 | 69345 | 02391 | 32647 | 27845 | 92017 | 82401 | 95342 | 02538 | 15230 | 60125 | 97813 | 57493 | 27386 | 13450 |
| | 70963 | 17820 | 68574 | 85091 | 16093 | 64538 | 93756 | 01786 | 67419 | 84679 | 84937 | 04625 | 20681 | 90514 | 97286 |
| 195 | 59816 | 13897 | 31250 | 32940 | 28754 | 48721 | 31748 | 43960 | 93028 | 93802 | 26039 | 47605 | 84961 | 39274 | 13064 |
| | 47023 | 45260 | 47896 | 68751 | 61039 | 63905 | 96250 | 78521 | 54176 | 47651 | 78145 | 81239 | 75203 | 86105 | 58927 |
| 210 | 03957 | 87529 | 60895 | 27615 | 21875 | 74590 | 90758 | 29064 | 57408 | 28947 | 69210 | 89710 | 53076 | 10578 | 36809 |
| | 84612 | 61403 | 23714 | 40839 | 69403 | 31286 | 34621 | 87513 | 63219 | 16053 | 37485 | 54236 | 92184 | 34269 | 14275 |
| 225 | 69012 | 15203 | 94652 | 47395 | 93605 | 43895 | 60274 | 40519 | 06384 | 27149 | 58302 | 76523 | 89631 | 18594 | 41059 |
| | 74385 | 47689 | 18073 | 01628 | 72148 | 20167 | 18359 | 28736 | 95721 | 30586 | 41679 | 91408 | 07524 | 07236 | 62738 |
| 240 | 57693 | 32816 | 87162 | 61348 | 15237 | 17620 | 17096 | 50139 | 56192 | 31970 | 32781 | 79358 | 23695 | 13864 | 24538 |
| | 48012 | 50749 | 45930 | 72509 | 86094 | 98543 | 34258 | 74286 | 73048 | 28564 | 50469 | 42061 | 04817 | 07529 | 60917 |
| 255 | 69407 | 45982 | 71896 | 32891 | 67984 | 93618 | 54867 | 28730 | 12674 | 84715 | 74269 | 28940 | 50628 | 92810 | 49168 |
| | 52138 | 70316 | 20453 | 64057 | 01325 | 25470 | 30912 | 51946 | 03895 | 23069 | 03851 | 65317 | 39417 | 34576 | 53270 |
| 270 | 20531 | 59762 | 71692 | 25684 | 39425 | 90132 | 14379 | 16720 | 16732 | 07423 | 71893 | 48307 | 64192 | 76915 | 79431 |
| | 98746 | 81340 | 84305 | 17309 | 86107 | 68547 | 56802 | 45983 | 84590 | 89165 | 02546 | 15269 | 85307 | 84023 | 05628 |
| 285 | 03124 | 41920 | 20834 | 61239 | 45723 | 12539 | 76530 | 38720 | 23697 | 19875 | 31256 | 83914 | 79064 | 74296 | 71326 |
| | 58967 | 73865 | 69175 | 80547 | 06981 | 04768 | 84291 | 19564 | 58104 | 06324 | 70984 | 20657 | 15832 | 53801 | 04859 |
| 300 | 36574 | 78296 | 04258 | 92407 | 82051 | 14928 | 78423 | 92530 | 85791 | 96513 | 49580 | 59671 | 29413 | 01534 | 19532 |
| | 90182 | 13045 | 37169 | 38651 | 73469 | 63570 | 01659 | 84671 | 34260 | 42807 | 72163 | 40823 | 05786 | 72986 | 67804 |
| 315 | 42689 | 70964 | 36297 | 68294 | 93784 | 31920 | 03621 | 27610 | 95274 | 46982 | 03529 | 35964 | 30176 | 46087 | 46208 |
| | 71503 | 81235 | 58140 | 30715 | 20651 | 86475 | 57948 | 35948 | 63081 | 31075 | 18764 | 21087 | 98254 | 19532 | 37159 |
| 330 | 35709 | 90582 | 01382 | 81564 | 07952 | 97462 | 90584 | 52048 | 50176 | 40791 | 28059 | 49650 | 91023 | 84723 | 34820 |
| | 41286 | 64371 | 57496 | 02397 | 18436 | 35801 | 21736 | 63197 | 43928 | 65328 | 17634 | 78312 | 65478 | 96051 | 96751 |
| 345 | 50182 | 61045 | 68974 | 54673 | 43782 | 48517 | 68975 | 29783 | 48591 | 08793 | 41986 | 32607 | 30912 | 07253 | 89536 |
| | 43967 | 79328 | 13205 | 91208 | 65091 | 06329 | 13420 | 54016 | 76032 | 54216 | 57023 | 95481 | 45867 | 61984 | 74201 |
| 360 | 40756 | 86974 | 43075 | 12690 | 06234 | 42905 | 60351 | 56148 | 94237 | 31056 | 01274 | 91724 | 59136 | 51082 | 89132 |
| | 38921 | 35021 | 89621 | 83457 | 78915 | 31786 | 78429 | 73092 | 18065 | 82794 | 65398 | 36508 | 07842 | 34679 | 47056 |

## TABLE XXXIII1. RANDOM PERMUTATIONS OF 10 NUMBERS—*continued*

| | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | 1 | 2 | 3 | 4 |
| 375 | 10759 | 51908 | 64982 | 87132 | 43718 | 03218 | 72186 | 62014 | 64389 | 84105 | 40735 | 40973 | 18329 | 31294 | 16402 |
| | 26348 | 42763 | 31075 | 94560 | 90256 | 94567 | 43059 | 87395 | 10572 | 37629 | 69218 | 26185 | 07645 | 76805 | 75398 |
| 390 | 63850 | 13069 | 57238 | 47958 | 41952 | 10798 | 43751 | 13508 | 32108 | 40913 | 27538 | 72104 | 57214 | 47068 | 06135 |
| | 94721 | 82457 | 16094 | 16302 | 08673 | 46352 | 86209 | 74269 | 47596 | 68725 | 04691 | 69358 | 06398 | 92135 | 78942 |
| 405 | 20459 | 59746 | 04659 | 46179 | 08913 | 13694 | 04398 | 47021 | 28937 | 50273 | 35410 | 82947 | 23841 | 10748 | 91385 |
| | 31786 | 18023 | 82137 | 23850 | 75624 | 02758 | 15267 | 38956 | 50461 | 94618 | 79862 | 01356 | 07596 | 52936 | 70246 |
| 420 | 51048 | 81502 | 98734 | 34015 | 79162 | 37298 | 94351 | 90485 | 42065 | 85290 | 31547 | 19864 | 28175 | 27319 | 96780 |
| | 32967 | 43679 | 21650 | 69278 | 83540 | 45160 | 06287 | 36721 | 91387 | 13674 | 06928 | 27530 | 64039 | 54806 | 12543 |
| 435 | 74302 | 34521 | 58640 | 87429 | 29536 | 68543 | 05179 | 75891 | 80574 | 19462 | 81537 | 45270 | 31270 | 08451 | 34295 |
| | 89561 | 96087 | 91732 | 36501 | 14870 | 79120 | 68243 | 36024 | 29316 | 50873 | 06294 | 16983 | 96854 | 72369 | 76801 |
| 450 | 86104 | 61943 | 12950 | 13045 | 81497 | 41092 | 83052 | 31854 | 48065 | 29485 | 86174 | 76485 | 79356 | 13809 | 86702 |
| | 72593 | 20857 | 68734 | 89267 | 02365 | 37685 | 61479 | 60792 | 32719 | 36710 | 25039 | 02913 | 14028 | 25647 | 35149 |
| 465 | 06853 | 48592 | 10835 | 42765 | 46792 | 70156 | 34876 | 02359 | 54736 | 08453 | 02967 | 54063 | 45139 | 37854 | 76014 |
| | 12749 | 61307 | 64297 | 38091 | 81530 | 23498 | 50912 | 68147 | 01289 | 21967 | 45183 | 91872 | 80726 | 61920 | 25893 |
| 480 | 87912 | 70324 | 69037 | 30815 | 51673 | 24897 | 14590 | 08297 | 95217 | 36581 | 14360 | 01829 | 50324 | 68271 | 49172 |
| | 03546 | 61985 | 54281 | 24967 | 82094 | 61503 | 73862 | 31645 | 48360 | 09427 | 29587 | 36475 | 69187 | 39540 | 03586 |
| 495 | 60294 | 25316 | 24936 | 48193 | 74318 | 78629 | 31609 | 92045 | 50729 | 34792 | 42795 | 56780 | 87056 | 87204 | 39208 |
| | 83175 | 80974 | 70815 | 57602 | 52609 | 41530 | 85427 | 68731 | 81643 | 60518 | 13608 | 21493 | 13924 | 56391 | 54761 |
| 510 | 67943 | 94068 | 91452 | 97510 | 17362 | 57093 | 83109 | 27364 | 96237 | 21395 | 87159 | 27138 | 32105 | 35067 | 51298 |
| | 52108 | 57132 | 60837 | 42386 | 49805 | 62418 | 27645 | 95180 | 51480 | 64078 | 02463 | 69540 | 48796 | 89241 | 03746 |
| 525 | 18506 | 61842 | 73260 | 86702 | 46293 | 67039 | 97625 | 17304 | 18456 | 50291 | 60784 | 21074 | 13520 | 47518 | 36795 |
| | 97342 | 09357 | 81954 | 19453 | 80751 | 21458 | 01843 | 98562 | 07239 | 67834 | 31592 | 83695 | 46978 | 30296 | 10428 |
| 540 | 08613 | 93286 | 47801 | 32769 | 58296 | 39021 | 10894 | 61238 | 72103 | 71943 | 56384 | 42136 | 73610 | 79318 | 94513 |
| | 95742 | 50471 | 56239 | 85014 | 70143 | 45678 | 57326 | 54079 | 96854 | 02568 | 01792 | 08795 | 52498 | 20465 | 27680 |
| 555 | 61270 | 80264 | 52689 | 05479 | 38512 | 54602 | 93052 | 51406 | 51342 | 78190 | 51937 | 40615 | 76238 | 87935 | 69048 |
| | 94835 | 95173 | 70314 | 86231 | 49076 | 81937 | 17468 | 27938 | 07689 | 36425 | 68420 | 83972 | 45190 | 60142 | 73215 |
| 570 | 54901 | 81326 | 58063 | 94053 | 36978 | 93048 | 90168 | 76102 | 60275 | 14935 | 04385 | 82453 | 08941 | 24385 | 74513 |
| | 86372 | 57490 | 24917 | 62781 | 50124 | 26715 | 74523 | 34958 | 93814 | 27860 | 91627 | 90671 | 25367 | 79610 | 98260 |
| 585 | 43027 | 59671 | 09746 | 20837 | 12436 | 87543 | 16482 | 75294 | 47038 | 42183 | 32509 | 06952 | 27589 | 35012 | 60287 |
| | 69815 | 80324 | 52813 | 49561 | 58790 | 16029 | 90753 | 30186 | 29516 | 56970 | 87614 | 38174 | 01436 | 98674 | 34591 |
| 600 | 12487 | 73960 | 09485 | 26340 | 75402 | 03279 | 17932 | 53160 | 73924 | 96823 | 52134 | 18532 | 49027 | 57218 | 14903 |
| | 56039 | 12458 | 62731 | 51897 | 18369 | 46851 | 56804 | 78294 | 15068 | 04517 | 08769 | 47960 | 58163 | 43906 | 85276 |
| 615 | 40928 | 47028 | 46120 | 63172 | 61283 | 15409 | 49752 | 85172 | 03126 | 13894 | 35980 | 34078 | 27509 | 95243 | 90746 |
| | 51376 | 31965 | 75398 | 84950 | 45907 | 83672 | 36801 | 46093 | 89475 | 67250 | 17426 | 16259 | 14863 | 78016 | 52138 |
| 630 | 08564 | 57064 | 53692 | 46273 | 87014 | 70256 | 12094 | 40986 | 86274 | 56308 | 90786 | 10928 | 96320 | 25634 | 26051 |
| | 79123 | 31298 | 74081 | 18590 | 59632 | 34981 | 63785 | 57132 | 05391 | 97142 | 21435 | 73654 | 48571 | 09781 | 39874 |
| 645 | 45382 | 96438 | 87094 | 97248 | 09473 | 76120 | 92867 | 91854 | 72196 | 08164 | 83497 | 41035 | 08725 | 74629 | 96401 |
| | 90176 | 50271 | 63215 | 06513 | 52618 | 85349 | 05314 | 20637 | 40835 | 92573 | 15620 | 28697 | 36914 | 53108 | 85237 |
| 660 | 01647 | 52768 | 81069 | 83214 | 56798 | 26389 | 03915 | 02863 | 79316 | 54760 | 28397 | 23751 | 48205 | 03721 | 54806 |
| | 35289 | 34019 | 32745 | 59760 | 14230 | 05471 | 47628 | 91574 | 84520 | 21389 | 06154 | 69084 | 13697 | 86594 | 29173 |
| 675 | 47521 | 89230 | 95326 | 08697 | 20185 | 76951 | 70563 | 19368 | 85093 | 85736 | 82094 | 54672 | 26459 | 23541 | 83602 |
| | 09368 | 45761 | 14087 | 53124 | 76349 | 38204 | 81294 | 24570 | 24176 | 02419 | 56137 | 91038 | 30187 | 70968 | 19547 |
| 690 | 07615 | 81049 | 54628 | 61394 | 28097 | 47302 | 47235 | 81075 | 90841 | 63914 | 96820 | 86312 | 86921 | 78610 | 92605 |
| | 92843 | 27653 | 93071 | 28507 | 31546 | 89165 | 86901 | 29346 | 62573 | 02857 | 34715 | 59704 | 50734 | 93245 | 47831 |
| 705 | 85390 | 02581 | 43582 | 63014 | 76039 | 92063 | 38045 | 63781 | 54231 | 85490 | 92183 | 50239 | 12497 | 97208 | 45312 |
| | 24167 | 73694 | 60971 | 92785 | 41528 | 71845 | 97261 | 05924 | 79806 | 73261 | 56047 | 61487 | 08563 | 63514 | 68970 |
| 720 | 06591 | 24503 | 06538 | 96532 | 73256 | 47510 | 30421 | 78439 | 87591 | 12087 | 15897 | 38795 | 92130 | 93825 | 63025 |
| | 74283 | 17986 | 41297 | 41087 | 40891 | 23869 | 79856 | 50126 | 02463 | 96534 | 04326 | 06241 | 67584 | 76401 | 89174 |
| 735 | 73502 | 62893 | 81025 | 93681 | 05681 | 86431 | 21863 | 73920 | 81930 | 53829 | 35492 | 78532 | 52134 | 40175 | 73854 |
| | 46981 | 05714 | 63479 | 70542 | 32497 | 50927 | 70495 | 41586 | 54726 | 16074 | 70816 | 96140 | 86790 | 32869 | 20691 |

# TABLE XXXIII2. RANDOM PERMUTATIONS OF 20 NUMBERS

|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 15 6 8 5<br>10 17 12 11 9<br>2 16 18 0 3<br>14 7 4 13 19 | 2 13 7 4 10<br>18 19 9 3 15<br>11 1 6 12 0<br>17 5 14 8 16 | 11 9 10 1 6<br>8 5 14 2 16<br>0 7 15 4 17<br>3 18 13 19 12 | 10 2 12 13 9<br>0 14 7 4 8<br>5 15 6 3 19<br>16 1 17 11 18 | 7 6 10 0 1<br>19 8 11 17 5<br>12 16 9 2 4<br>13 15 18 3 14 | 16 19 14 8 18<br>0 12 13 3 4<br>6 10 2 5 7<br>1 17 9 15 11 | 1 15 3 10 17<br>18 7 2 8 16<br>0 11 5 12 9<br>14 19 13 6 4 | 3 16 0 15 9<br>17 8 6 14 7<br>5 12 1 10 13<br>11 4 19 18 2 | 14 19 12 17 10<br>11 13 7 5 3<br>0 2 6 8 18<br>1 16 9 4 15 | 13 17 11 12 14<br>10 4 8 6 9<br>1 0 3 18 16<br>2 7 19 5 15 |
| 1 | 11 19 2 15 1<br>7 14 5 4 3<br>16 9 18 0 12<br>8 13 10 17 6 | 3 16 13 7 12<br>6 18 8 2 19<br>9 1 11 10 5<br>0 17 14 15 4 | 3 15 5 0 11<br>7 12 13 18 4<br>8 10 17 6 16<br>9 1 14 19 2 | 12 9 16 10 1<br>6 13 2 19 5<br>18 4 0 8 7<br>11 17 14 15 3 | 2 16 6 18 19<br>0 7 10 15 3<br>8 14 5 9 13<br>1 11 4 12 17 | 11 14 10 13 5<br>0 17 4 9 7<br>8 12 16 6 15<br>3 18 19 2 1 | 4 14 3 10 1<br>8 11 18 9 6<br>16 2 15 12 13<br>19 7 17 5 0 | 6 4 19 14 17<br>16 7 3 11 2<br>10 8 0 9 5<br>1 18 15 12 13 | 18 17 9 19 6<br>5 14 13 1 4<br>7 15 12 3 2<br>0 10 8 11 16 | 0 13 18 16 2<br>11 6 12 14 17<br>8 15 1 19 10<br>7 5 4 9 3 |
| 2 | 18 8 11 9 15<br>4 5 14 3 0<br>7 10 16 12 6<br>19 2 1 13 17 | 19 13 4 1 11<br>2 0 7 16 18<br>5 17 6 8 14<br>9 10 12 15 3 | 18 6 3 17 11<br>4 14 9 10 16<br>1 12 2 8 19<br>15 5 0 13 7 | 3 1 14 12 0<br>8 2 10 13 18<br>17 7 11 16 4<br>5 15 9 19 6 | 2 17 0 7 19<br>14 9 12 11 8<br>6 3 16 13 5<br>15 10 1 4 18 | 1 2 10 9 12<br>6 11 15 0 17<br>16 8 4 18 5<br>13 3 14 19 7 | 17 7 4 3 19<br>10 0 13 11 14<br>8 9 5 2 18<br>16 12 6 15 1 | 13 0 9 5 17<br>15 11 1 12 7<br>10 14 4 18 16<br>3 2 19 6 8 | 1 12 9 3 16<br>5 17 0 7 13<br>18 4 19 11 15<br>8 6 2 10 14 | 0 11 8 4 12<br>7 2 15 3 18<br>10 13 6 16 5<br>9 17 19 1 14 |
| 3 | 4 0 1 12 7<br>3 14 5 15 6<br>8 17 11 19 10<br>9 18 16 13 2 | 12 9 0 5 19<br>7 3 4 6 1<br>8 13 2 18 16<br>11 17 14 10 15 | 15 16 2 5 19<br>7 8 3 4 12<br>6 1 0 17 10<br>9 13 18 11 14 | 5 14 16 11 6<br>15 13 3 1 17<br>19 2 7 8 18<br>12 9 0 10 4 | 11 7 5 12 2<br>13 19 15 14 3<br>9 10 8 6 1<br>16 0 4 18 17 | 15 11 8 5 3<br>1 9 18 0 14<br>4 12 16 6 17<br>19 10 2 7 13 | 18 14 8 19 15<br>6 4 3 7 1<br>0 13 12 2 10<br>17 9 16 5 11 | 8 16 19 0 4<br>13 5 11 14 6<br>1 18 17 7 12<br>15 10 3 2 9 | 13 8 5 1 14<br>18 11 2 4 12<br>10 9 3 15 6<br>17 16 0 7 19 | 1 11 3 17 7<br>9 16 0 6 5<br>4 18 8 15 2<br>13 14 12 19 10 |
| 4 | 0 2 5 1 18<br>15 6 14 12 8<br>13 16 17 3 19<br>10 9 7 4 11 | 14 16 19 7 8<br>1 11 9 3 4<br>10 6 5 17 13<br>12 15 2 18 0 | 14 6 8 1 19<br>10 7 12 13 5<br>17 0 15 18 4<br>3 9 11 16 2 | 5 2 3 14 13<br>18 9 0 12 16<br>10 6 15 17 7<br>8 19 1 11 4 | 18 9 2 12 16<br>3 10 7 8 13<br>15 14 17 0 6<br>11 19 4 1 5 | 0 1 18 4 15<br>17 8 13 6 16<br>14 9 12 19 3<br>11 5 2 10 7 | 13 14 0 2 11<br>3 19 18 1 6<br>12 10 4 15 17<br>5 9 16 7 8 | 0 14 5 12 13<br>9 16 7 6 4<br>3 19 18 10 1<br>2 17 15 11 8 | 0 6 18 7 16<br>8 11 1 10 15<br>3 17 5 2 13<br>14 4 9 12 19 | 7 9 19 11 5<br>18 16 2 12 6<br>14 13 1 17 0<br>3 15 4 10 8 |
| 5 | 6 2 8 4 14<br>18 1 10 16 9<br>7 15 5 3 12<br>13 0 11 19 17 | 7 16 6 2 1<br>5 8 17 0 12<br>9 18 10 3 11<br>15 19 4 14 13 | 14 8 3 19 4<br>9 15 0 11 5<br>1 6 2 18 7<br>13 16 17 10 12 | 7 16 11 19 17<br>5 14 18 9 3<br>0 10 4 8 12<br>15 13 1 6 2 | 2 11 0 1 7<br>19 13 18 8 6<br>9 15 3 17 16<br>4 12 5 10 14 | 10 4 12 5 19<br>8 7 14 2 18<br>0 3 9 15 13<br>6 17 1 16 11 | 4 13 11 9 2<br>10 15 16 17 6<br>19 8 14 3 7<br>18 0 1 12 5 | 1 6 19 16 11<br>17 5 14 9 13<br>10 18 8 2 15<br>3 0 7 12 4 | 5 3 11 9 0<br>2 7 12 15 10<br>8 6 19 1 14<br>16 17 13 4 18 | 3 6 7 9 1<br>11 16 4 12 10<br>17 8 2 13 19<br>0 5 14 18 15 |
| 6 | 19 13 17 0 9<br>3 5 1 7 14<br>8 6 15 10 4<br>2 11 12 18 16 | 8 7 10 5 4<br>9 1 15 6 12<br>13 18 19 2 11<br>14 17 16 0 3 | 12 15 5 17 9<br>6 1 18 14 4<br>2 16 19 7 11<br>0 8 13 3 10 | 17 19 0 14 11<br>3 2 18 8 16<br>4 10 6 12 1<br>7 15 5 9 13 | 4 10 7 19 2<br>0 16 3 8 6<br>14 11 17 18 15<br>12 9 13 5 1 | 17 10 3 0 14<br>2 9 18 19 1<br>8 15 6 11 12<br>4 5 13 16 7 | 19 9 4 17 1<br>10 14 15 18 3<br>6 16 5 13 7<br>0 8 11 12 2 | 11 7 17 3 10<br>14 1 2 19 6<br>15 0 5 9 8<br>12 16 13 4 18 | 5 18 1 13 8<br>11 0 6 3 19<br>17 9 16 10 15<br>4 14 12 2 7 | 13 10 18 16 19<br>1 11 14 8 4<br>0 6 5 9 17<br>15 2 12 3 7 |
| 7 | 3 1 17 13 19<br>5 18 12 15 2<br>0 9 16 4 10<br>11 8 14 7 6 | 14 1 12 6 8<br>18 17 2 3 5<br>15 19 13 16 4<br>0 7 10 9 11 | 16 12 6 7 1<br>15 5 18 0 17<br>8 10 19 3 11<br>2 4 9 13 14 | 2 18 3 8 6<br>4 7 14 0 10<br>19 1 9 11 13<br>15 12 16 5 17 | 11 1 6 8 9<br>18 12 15 0 19<br>13 2 10 14 5<br>4 3 7 16 17 | 6 1 10 11 15<br>8 7 2 14 4<br>13 17 0 9 19<br>16 3 12 18 5 | 2 11 10 9 8<br>1 13 18 3 16<br>12 14 17 6 19<br>5 15 7 4 0 | 17 7 6 15 19<br>18 0 16 4 3<br>10 2 9 5 12<br>14 8 1 11 13 | 10 0 14 3 19<br>13 18 1 8 2<br>12 11 15 9 7<br>4 5 16 6 17 | 1 3 13 9 14<br>6 12 4 18 11<br>16 8 2 17 7<br>0 19 15 5 10 |
| 8 | 19 16 6 8 15<br>2 10 4 14 18<br>9 0 12 5 13<br>11 1 7 3 17 | 10 11 9 3 12<br>14 2 4 17 0<br>15 5 16 18 6<br>19 7 13 8 1 | 13 6 4 8 5<br>1 2 9 14 11<br>3 12 0 15 7<br>16 10 17 18 19 | 13 7 6 1 2<br>8 9 0 12 11<br>19 14 5 4 3<br>15 17 10 18 16 | 12 15 13 5 3<br>4 9 18 16 7<br>2 10 0 17 1<br>6 11 19 8 14 | 10 15 0 11 9<br>3 16 13 5 2<br>18 19 14 4 12<br>17 6 1 7 8 | 19 18 7 11 17<br>4 5 0 16 2<br>12 1 6 10 3<br>13 14 15 9 8 | 13 14 10 6 18<br>2 12 4 1 3<br>17 19 9 16 15<br>11 8 7 0 5 | 18 10 17 7 1<br>12 13 4 8 9<br>14 16 3 15 5<br>2 0 19 11 6 | 8 10 6 18 11<br>9 17 13 19 1<br>15 14 4 16 3<br>2 5 12 0 7 |
| 9 | 2 7 18 0 17<br>3 11 1 16 15<br>13 19 12 6 8<br>9 4 10 14 5 | 5 6 13 15 16<br>7 9 19 10 4<br>3 1 12 18 8<br>17 11 2 0 14 | 4 15 9 3 10<br>13 8 6 14 12<br>0 16 7 5 2<br>18 17 11 1 19 | 13 18 19 4 3<br>9 16 14 5 0<br>17 1 11 15 10<br>2 6 8 12 7 | 9 17 15 11 10<br>6 1 4 16 7<br>19 5 14 3 0<br>12 2 8 13 18 | 13 10 11 3 14<br>7 9 0 12 5<br>18 19 17 8 6<br>2 16 15 4 1 | 6 13 1 19 12<br>4 7 15 8 3<br>2 18 14 16 5<br>9 0 17 11 10 | 9 16 2 8 6<br>19 14 5 15 10<br>1 4 13 18 11<br>3 17 12 7 0 | 13 2 1 9 5<br>8 6 14 4 16<br>0 18 11 17 12<br>19 15 10 3 7 | 3 15 10 5 1<br>16 8 12 13 0<br>7 9 4 2 11<br>18 6 17 14 19 |

## TABLE XXXIII2. RANDOM PERMUTATIONS OF 20 NUMBERS—*continued*

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 17 7 3 13 12 | 17 1 12 3 2 | 2 9 16 4 6 | 3 4 10 13 16 | 6 17 13 19 0 | 7 13 17 11 1 | 14 10 16 4 7 | 10 12 15 17 5 | 15 2 3 12 1 | 0 9 11 8 6 |
| | 16 15 1 10 14 | 13 10 14 15 16 | 3 5 13 10 17 | 14 0 18 5 9 | 12 15 18 9 3 | 0 4 14 6 19 | 8 11 13 19 15 | 2 9 4 1 8 | 13 4 10 18 19 | 7 18 15 12 1 |
| | 2 19 8 4 5 | 5 6 7 9 19 | 18 8 7 12 1 | 17 15 12 11 6 | 14 2 1 7 4 | 8 5 2 16 18 | 0 9 18 6 3 | 13 0 14 16 7 | 16 9 7 8 0 | 19 13 10 17 5 |
| | 0 9 11 6 18 | 18 0 11 8 4 | 15 0 11 14 19 | 7 1 19 2 8 | 10 16 5 8 11 | 12 10 15 3 9 | 12 1 5 2 17 | 11 6 19 18 3 | 14 17 11 5 6 | 14 16 2 4 3 |
| 11 | 16 1 17 15 11 | 6 16 1 18 17 | 8 16 11 3 9 | 4 10 15 16 12 | 6 13 8 12 1 | 1 7 2 9 4 | 14 3 4 1 10 | 10 16 0 15 3 | 1 19 8 6 13 | 0 14 18 11 5 |
| | 8 6 14 7 10 | 19 9 10 3 4 | 12 0 4 2 13 | 1 9 8 6 5 | 9 17 5 10 11 | 10 0 17 19 3 | 2 0 12 17 19 | 7 8 19 13 17 | 17 3 16 15 18 | 3 19 13 12 4 |
| | 9 0 3 5 4 | 13 0 12 11 2 | 6 5 19 14 7 | 18 7 14 13 0 | 18 14 7 4 0 | 8 16 13 5 11 | 15 9 16 7 18 | 5 1 9 2 4 | 4 5 14 0 12 | 10 7 2 15 1 |
| | 19 18 2 12 13 | 7 8 5 14 15 | 10 18 1 15 17 | 3 19 11 17 2 | 3 19 15 2 16 | 12 6 14 15 18 | 11 13 5 6 8 | 18 14 12 6 11 | 2 10 7 11 9 | 6 16 9 17 8 |
| 12 | 4 13 10 18 14 | 6 10 8 11 9 | 3 7 4 2 14 | 15 17 12 6 0 | 8 0 13 12 18 | 0 16 6 4 18 | 17 11 6 16 15 | 1 19 0 4 8 | 17 19 5 8 16 | 18 19 17 6 10 |
| | 2 5 0 1 11 | 16 13 5 0 3 | 1 5 13 18 15 | 14 4 16 19 5 | 2 3 19 9 5 | 14 5 1 2 3 | 13 1 18 7 19 | 16 6 18 14 17 | 3 7 18 4 0 | 3 1 0 11 16 |
| | 19 7 3 16 6 | 12 2 14 4 7 | 0 9 10 11 17 | 3 11 13 7 2 | 7 11 14 10 15 | 15 13 19 7 12 | 5 3 9 2 12 | 2 12 13 7 9 | 13 1 14 6 11 | 2 13 5 4 8 |
| | 15 12 9 8 17 | 17 18 1 15 19 | 19 16 6 12 8 | 10 9 8 18 1 | 6 16 1 17 4 | 11 8 17 9 10 | 14 4 0 8 10 | 15 5 3 11 10 | 12 15 2 10 9 | 7 14 9 12 15 |
| 13 | 11 1 5 0 14 | 0 7 17 19 3 | 16 2 18 15 1 | 10 0 1 17 11 | 19 6 18 5 16 | 15 11 5 14 8 | 11 14 15 18 5 | 16 18 7 17 11 | 7 19 12 2 13 | 8 13 4 0 3 |
| | 19 9 2 4 8 | 11 12 13 10 9 | 9 4 5 3 6 | 16 19 7 6 3 | 10 15 13 11 7 | 10 1 12 3 19 | 13 7 3 8 10 | 13 9 10 19 1 | 10 15 3 1 17 | 5 10 7 9 17 |
| | 17 10 6 12 15 | 5 15 6 8 1 | 8 13 17 14 7 | 15 8 5 4 2 | 12 0 14 1 4 | 4 18 7 13 0 | 16 4 1 0 19 | 0 12 8 6 14 | 9 16 5 4 14 | 15 11 14 2 16 |
| | 16 13 7 18 3 | 14 18 2 4 16 | 11 19 10 0 12 | 14 18 13 9 12 | 3 9 8 2 17 | 9 6 17 16 2 | 12 6 9 2 17 | 5 4 2 15 3 | 0 6 8 18 11 | 6 19 18 1 12 |
| 14 | 18 5 12 3 6 | 11 16 1 8 3 | 16 6 0 8 18 | 13 17 14 15 0 | 9 5 16 4 18 | 2 7 13 3 12 | 8 0 3 15 10 | 8 7 5 15 10 | 1 15 0 13 11 | 18 2 5 6 19 |
| | 1 2 11 15 7 | 18 14 19 9 13 | 17 5 7 19 10 | 8 12 2 16 11 | 19 1 11 17 10 | 10 17 5 11 8 | 16 6 2 14 9 | 1 9 4 6 0 | 8 17 3 10 12 | 8 13 9 10 4 |
| | 14 4 10 9 17 | 12 2 5 17 4 | 4 9 12 3 2 | 9 18 6 4 10 | 12 14 2 13 6 | 14 9 19 16 6 | 17 18 4 19 11 | 11 19 13 17 2 | 5 4 6 18 7 | 12 15 7 3 1 |
| | 16 8 0 13 19 | 7 0 10 6 15 | 14 13 1 15 11 | 19 3 7 1 5 | 0 7 8 3 15 | 4 15 18 1 0 | 1 13 12 5 7 | 12 16 18 14 3 | 19 14 9 16 2 | 16 0 17 14 11 |
| 15 | 5 1 11 7 10 | 2 14 4 6 18 | 18 4 17 16 5 | 15 0 1 14 4 | 14 13 16 7 12 | 19 5 15 7 9 | 19 8 3 1 12 | 14 6 8 7 18 | 7 17 14 13 4 | 18 10 15 16 0 |
| | 14 16 6 15 12 | 8 10 1 13 19 | 9 6 1 19 11 | 8 6 2 10 18 | 6 2 8 10 19 | 17 18 13 11 16 | 4 17 13 18 2 | 9 5 19 17 2 | 12 8 19 1 3 | 13 14 17 19 9 |
| | 17 3 4 2 0 | 17 16 12 3 11 | 7 2 13 15 8 | 12 9 7 16 3 | 3 1 11 9 18 | 14 3 6 10 1 | 10 16 9 7 0 | 1 4 12 11 16 | 0 10 5 11 2 | 6 2 3 12 4 |
| | 18 13 9 19 8 | 15 0 7 5 9 | 14 3 12 0 10 | 5 13 11 17 19 | 17 5 15 4 0 | 4 12 8 0 2 | 14 15 6 11 5 | 15 0 13 10 3 | 9 16 6 15 18 | 11 8 1 7 5 |
| 16 | 15 14 0 13 3 | 19 4 12 3 6 | 6 15 9 10 12 | 8 7 0 17 15 | 12 18 17 3 5 | 7 14 0 2 3 | 12 9 14 13 7 | 13 1 11 19 4 | 18 3 4 0 1 | 13 19 3 7 4 |
| | 4 17 2 11 16 | 14 11 13 18 17 | 4 19 14 7 5 | 10 13 1 2 6 | 15 19 14 7 16 | 1 11 9 5 16 | 2 1 11 15 0 | 2 16 5 9 3 | 8 2 14 17 7 | 16 10 9 14 15 |
| | 10 18 12 6 8 | 0 7 10 1 8 | 17 3 16 2 18 | 19 4 9 12 11 | 8 4 13 6 11 | 17 13 10 12 18 | 16 3 19 17 4 | 15 7 10 8 18 | 15 13 6 19 12 | 6 2 11 8 18 |
| | 7 9 1 19 5 | 9 2 16 5 15 | 13 8 1 0 11 | 3 18 16 5 14 | 1 2 9 0 10 | 4 19 8 6 15 | 18 5 8 6 10 | 0 17 6 12 14 | 10 16 9 11 5 | 1 0 5 17 12 |
| 17 | 4 7 18 3 8 | 1 15 16 18 13 | 2 7 12 4 0 | 14 12 15 17 0 | 16 17 6 5 11 | 7 2 4 1 9 | 16 8 14 6 19 | 2 16 9 6 12 | 4 16 2 7 18 | 19 3 6 2 16 |
| | 13 15 5 6 0 | 2 8 6 10 3 | 9 15 19 13 5 | 8 18 4 2 16 | 10 4 15 7 13 | 15 19 11 16 14 | 2 5 11 7 0 | 8 15 10 14 18 | 0 17 15 10 6 | 14 15 12 5 4 |
| | 10 2 9 17 12 | 17 5 19 4 0 | 18 16 10 8 6 | 13 1 10 7 3 | 14 1 3 2 0 | 0 10 18 17 13 | 15 17 4 3 9 | 5 13 19 1 4 | 1 12 5 14 19 | 9 7 11 10 8 |
| | 11 14 1 19 16 | 11 9 7 14 12 | 3 11 17 1 14 | 9 19 6 5 11 | 18 19 8 9 12 | 5 8 12 6 3 | 1 12 10 18 13 | 7 3 0 11 17 | 3 13 9 8 11 | 18 13 17 1 0 |
| 18 | 7 10 16 12 1 | 19 0 16 14 8 | 14 11 12 2 5 | 4 0 15 5 9 | 11 18 17 4 2 | 5 14 11 16 1 | 5 3 16 2 0 | 16 7 6 8 1 | 0 13 4 11 14 | 14 9 16 15 1 |
| | 18 8 14 3 19 | 2 9 18 10 4 | 15 1 0 6 18 | 7 18 13 17 1 | 15 0 12 5 8 | 18 17 0 10 15 | 1 13 10 19 7 | 4 15 14 17 10 | 15 9 6 8 1 | 13 7 8 11 17 |
| | 13 17 9 5 15 | 5 6 13 1 11 | 8 16 13 10 3 | 19 10 14 11 2 | 14 7 19 1 10 | 2 13 6 8 9 | 18 14 12 17 6 | 9 12 19 0 11 | 3 17 19 10 2 | 5 18 3 10 12 |
| | 2 6 4 11 0 | 3 12 7 17 15 | 19 4 7 9 17 | 12 8 3 16 6 | 13 16 6 9 3 | 4 12 7 19 3 | 11 9 8 15 4 | 18 2 13 3 5 | 5 18 12 7 16 | 6 4 0 2 19 |
| 19 | 4 1 10 18 13 | 17 15 9 19 4 | 7 4 19 12 16 | 6 7 9 13 3 | 4 2 19 3 0 | 12 7 6 19 0 | 12 14 7 19 16 | 8 12 10 3 19 | 5 6 3 16 10 | 1 2 6 9 5 |
| | 0 8 14 16 12 | 7 18 12 3 14 | 5 0 9 3 10 | 0 12 8 17 18 | 9 13 7 14 5 | 2 14 8 16 18 | 4 8 18 10 2 | 11 4 6 5 13 | 19 1 4 12 13 | 17 19 15 14 13 |
| | 9 15 7 3 17 | 1 8 13 2 5 | 2 14 8 17 11 | 16 1 2 4 5 | 15 12 17 8 6 | 5 13 11 17 4 | 13 3 0 5 6 | 2 16 9 18 0 | 17 2 0 15 9 | 10 18 3 12 7 |
| | 2 6 5 19 11 | 0 10 6 16 11 | 18 13 15 1 6 | 15 10 19 14 11 | 10 1 18 16 11 | 9 15 10 3 1 | 17 11 1 9 15 | 1 15 17 7 14 | 18 8 7 11 14 | 0 8 4 16 11 |

# TABLE XXXIV. CONSTANTS, WEIGHTS AND MEASURES, ETC.

## *Mathematical Constants*

$\pi = 3{\cdot}14159\ 26535\ 89793\ 23846$

$\log_{10}\pi = 0{\cdot}49714\ 98726\ 94133\ 85435$

$1/\sqrt{2\pi} = 0{\cdot}39894\ 22804\ 01432\ 67794$

$e = 2{\cdot}71828\ 18284\ 59045\ 23536$

$\log_{10}e = 0{\cdot}43429\ 44819\ 03251\ 82765$

$\log_e 10 = 2{\cdot}30258\ 50929\ 94045\ 68402$

$\gamma = 0{\cdot}57721\ 56649\ 01532\ 86061$

$e^{\gamma} = 1{\cdot}78107\ 24179\ 90197\ 98522$

$e^{-\gamma} = 0{\cdot}56145\ 94835\ 66885\ 16983$

## *Physical Constants*

Velocity of light = $3{\cdot}00 \times 10^{10}$ cm./sec. = 186,000 miles/sec. The first number is the ratio of electromagnetic to electrostatic unit of charge.

Ionic (electronic) charge ($e$) = $4{\cdot}77 \times 10^{-10}$ E.S.U. = $1{\cdot}59 \times 10^{-20}$ E.M.U. Mass of electron ($m_0$) = $9{\cdot}00 \times 10^{-28}$ g.

Planck's constant ($h$) = $6{\cdot}55 \times 10^{-27}$ erg secs.

Energy of gas molecule per 1° C. absolute = $2{\cdot}06 \times 10^{-16}$ ergs.

Number of molecules in 1 gm. molecule of gas = $6{\cdot}06 \times 10^{23}$.

Electrical charge of 1 gm. equivalent in electrolysis (Faraday) = $9{\cdot}65 \times 10^{3}$ E.M.U.

Gas constant ($R$) = $8{\cdot}315 \times 10^{7}$ c.g.s. ($pv = RT \times$ mass in gram molecules).

Constant of gravitation ($G$) = $6{\cdot}67 \times 10^{-8}$ c.g.s. ($f = Gmm'/r^2$).

Acceleration due to gravity ($g$). Teddington (N. 51° 28·1′): 981·181 cm./sec.$^2$ = 32·191 ft./sec.$^2$; Washington (38° 56·5′): 980·082; whence at lat. $\lambda$, height $H$ m.: $980{\cdot}618 - 2{\cdot}586 \cos 2\lambda + {\cdot}003 \cos 4\lambda, - {\cdot}0003086\,H$.

The earth. Equatorial radius = 3963·4 miles, polar radius = 3950·0 miles. 1° of Latitude = 68·70 mls. (equator), 69·41 mls. (pole). Mean density = 345 lb./ft.$^3$ = 5·517 g./cm.$^3$

Distance of sun = $93 \times 10^{6}$ miles. Distance of moon = 239,000 miles.

Solar radiation (Washington, D.C.) = ·035 cal./cm.$^2$/sec.

Velocity of sound (6° C.) = $3{\cdot}35 \times 10^{4}$ cm./sec. = 1100 ft./sec. Audible sound: Lower limit, 30 vibr./sec.; upper limit, 24,000-41,000 vibr./sec. Middle C of piano, 261 vibr./sec. Wave-lengths of visible light: 4 to $7 \times 10^{-5}$ cm.

| Temperature Equivalents | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Centigrade | −40 | −30 | −20 | −10 | 0 | 10 | 20 | 30 | 40 | 50 | **100** |
| Fahrenheit | −40 | −22 | −4 | 14 | **32** | 50 | 68 | 86 | 104 | 122 | **212** |
| Réaumur | −32 | −24 | −16 | −8 | 0 | 8 | 16 | 24 | 32 | 40 | **80** |

1° F. = $\frac{5}{9}$° C. 1° R. = $\frac{5}{4}$° C. Absolute zero of temperature = −273·1° C.

## *Mechanical and Electrical Units*

*Force.* $g$ dynes = 1 gm. weight, $g$ poundals = 1 lb. weight, where $g$ is acceleration due to gravity measured in appropriate units (cm./sec.$^2$ or ft./sec.$^2$). 1 poundal = $1{\cdot}3825 \times 10^{4}$ dynes.

*Pressure.* 1 atmosphere = 760 mm. (29·921 ins.) of mercury (0° C., Lat. 45°) = 33·899 ft. of water (4° C.) = 14·696 lb./in.$^2$ = $1{\cdot}01325 \times 10^{6}$ dynes/cm.$^2$ 1 bar = $10^{6}$ dynes/cm.$^2$ = ·98692 atmos. 1 lb./in.$^2$ = 70·307 (1/·014223) gm./cm.$^2$

*Work, Power, etc.* 1 erg = 1 dyne cm. 1 joule = $10^{7}$ ergs = 1 watt second. 1 foot pound = $1{\cdot}356 \times 10^{7}$ ergs. 1 horse power (H.P.) = 550 ft. lb. per sec. = 746 watts (continental H.P. = 736 watts). 1 calorie = heat to raise 1 gm. of water 1° C. at a specified temperature. 1 15° C. calorie = 1 mean calorie (0° C.-100° C.) = $4{\cdot}185 \times 10^{7}$ ergs. 1 British Thermal Unit (1 lb. water × 1° F.) = 252·0 calories = $2{\cdot}930 \times 10^{-4}$ kilowatt hours = $3{\cdot}929 \times 10^{-4}$ horse power hours = $10^{-5}$ gas therms. 1 Board of Trade Unit = 1 kilowatt hour.

*Electrical.* Relations between engineering units and electromagnetic units. Charge: coulomb = $10^{-1}$ E.M.U. Current: ampere = $10^{-1}$ E.M.U. Potential: volt = $10^{8}$ E.M.U. Resistance: ohm = $10^{9}$ E.M.U. Capacity: farad = $10^{-9}$ E.M.U. Inductance: henry = $10^{9}$ E.M.U. Flux: maxwell = $10^{0}$ E.M.U. Magnetic force: gauss = $10^{0}$ E.M.U.

## *Supposed Duration of Geologic Periods* (*E. W. Barnes*)

| | | Duration Million Years. |
|---|---|---|
| Tertiary or Cainozoic era<br>Age of Mammals | Pleistocene<br>Pliocene<br>Miocene<br>Oligocene<br>Eocene | 60 |
| Secondary or Mesozoic era<br>Age of reptiles | Cretaceous<br>Jurassic<br>Triassic | 85 |
| Newer Palæozoic era | Permian<br>Carboniferous<br>Devonian | 175 |
| Older Palæozoic era | Silurian<br>Ordovician<br>Cambrian | 250 |
| Archæozoic era . . . . . | | 300 |
| Eozoic era . . . . . | | 500 |

Possible age of planet 2000-4000 million years.

# TABLE XXXIV. CONSTANTS, WEIGHTS AND MEASURES, ETC.—*continued*

## *The Elements and their Atomic Weights*

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| H | Hydrogen | 1·0080 | Cr | Chromium | 52·01 | Ag | Silver | 107·880 | Yb | Ytterbium | 173·04 |
| He | Helium | 4·003 | Mn | Manganese | 54·93 | Cd | Cadmium | 112·41 | Lu | Lutetium | 174·99 |
| Li | Lithium | 6·940 | Fe | Iron | 55·85 | In | Indium | 114·76 | Hf | Hafnium | 178·6 |
| Be | Beryllium | 9·013 | Co | Cobalt | 58·94 | Sn | Tin | 118·70 | Ta | Tantalum | 180·88 |
| B | Boron | 10·82 | Ni | Nickel | 58·69 | Sb | Antimony | 121·76 | W | Tungsten | 183·92 |
| C | Carbon | 12·010 | Cu | Copper | 63·54 | Te | Tellurium | 127·61 | Re | Rhenium | 186·31 |
| N | Nitrogen | 14·008 | Zn | Zinc | 65·38 | I | Iodine | 126·91 | Os | Osmium | 190·2 |
| | Oxygen | 16·000 | Ga | Gallium | 69·72 | Xe | Xenon, | 131·3 | Ir | Iridium | 193·1 |
| | Fluorine | 19·00 | Ge | Germanium | 72·60 | Cs | Cesium | 132·91 | Pt | Platinum | 195·23 |
| e | Neon | 20·183 | As | Arsenic | 74·91 | Ba | Barium | 137·36 | Au | Gold | 197·2 |
| a | Sodium | 22·997 | Se | Selenium | 78·96 | La | Lanthanum | 138·92 | Hg | Mercury | 200·61 |
| g | Magnesium | 24·32 | Br | Bromine | 79·916 | Ce | Cerium | 140·13 | Tl | Thallium | 204·39 |
| | Aluminium | 26·98 | Kr | Krypton | 83·80 | Pr | Praseodymium | 140·92 | Pb | Lead | 207·21 |
| | Silicon | 28·09 | Rb | Rubidium | 85·48 | Nd | Neodymium | 144·27 | Bi | Bismuth | 209·00 |
| | Phosphorus | 30·975 | Sr | Strontium | 87·63 | Pm | Promethium | (145) | Po | Polonium | 210 |
| | Sulphur | 32·07 | Y | Yttrium | 88·92 | Sm | Samarium | 150·43 | At | Astatine | (210) |
| | Chlorine | 35·457 | Zr | Zirconium | 91·22 | Eu | Europium | 152·0 | Rn | Radon | 222 |
| | Argon | 39·944 | Nb | Niobium | 92·91 | Gd | Gadolinium | 156·9 | Fr | Francium | (223) |
| | Potassium | 39·100 | Mo | Molybdenum | 95·95 | Tb | Terbium | 159·2 | Ra | Radium | 226·05 |
| | Calcium | 40·08 | Tc | Technetium | (99) | Dy | Dysprosium | 162·46 | Ac | Actinium | 227 |
| | Scandium | 44·96 | Ru | Ruthenium | 101·7 | Ho | Holmium | 164·94 | Th | Thorium | 232·12 |
| | Titanium | 47·90 | Rh | Rhodium | 102·91 | Er | Erbium | 167·2 | Pa | Protoactinium | 231 |
| | Vanadium | 50·95 | Pd | Palladium | 106·7 | Tm | Thulium | 169·4 | U | Uranium | 238·07 |

## *Densities*

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| uminium | 2·70 | Iron | 7·86 | Platinum | 21·50 | Coal | 1·2-1·8 | White pine | ·4 - ·5 | Glass | 2·4 -4·5 |
| pper | 8·93 | Lead | 11·37 | Silver | 10·5 | Granite | 2·5-3 | Beech, oak | ·7 - ·9 | Sea water | 1·01-1·05 |
| iamond | 3·52 | Magnesium | 1·74 | Tin | 7·29 | Quartz | 2·66 | Ebony | 1·1 -1·3 | Loose snow | ·12 approx. |
| old | 19·32 | Mercury | 13·56 | Zinc | 7·1 | Slate | 2·5-2·7 | Cork | ·22 - ·26 | Soil | 1·6 approx. |

Unit density = 1 g. per cu. cm. = ·036128 lb. per cu. in.

## *Equation of Time*

| | | | | | |
|---|---|---|---|---|---|
| n. 1 + 3m. 11s. | Mar. 1 +12m. 34s. | May 1 −2m. 57s. | July 1 +3m.32s. | Sept. 16 − 5m. 6s. | Nov. 16 −15m.10s. |
| ,, 16 + 9 33 | ,, 16 + 8 51 | ,, 14 −3 49* | ,, 26 +6 18* | Oct. 1 −10 16 | Dec. 1 −10 56 |
| eb. 1 +13 37 | Apr. 1 + 4 1 | June 1 −2 27 | Aug. 16 +4 11 | ,, 16 −14 20 | ,, 12 − 6 15 |
| ,, 12 +14 25* | ,, 16 0 0 | ,, 15 0 0 | Sept. 1 0 0 | Nov. 3 −16 21* | ,, 25 0 0 |

When positive add the equation of time to apparent solar time to obtain mean solar time. The values given are mean values. Maxima and minima are marked with asterisks. 1 sidereal day = 23 h. 56 m. 4·09 s.

## *Saturated Vapour Pressure of Water (Millibars)*

| °C. | −50 | −40 | −30 | −20 | −10 | −5 | −2 |
|---|---|---|---|---|---|---|---|
| Ice | ·039 | ·129 | ·381 | 1·035 | 2·600 | 4·017 | 5·173 |
| ater | — | — | — | 1·270 | 2·865 | 4·217 | 5·274 |

| °C. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 6·105 | 6·567 | 7·058 | 7·579 | 8·134 | 8·723 | 9·350 | 10·02 | 10·73 | 11·48 |
| 10 | 12·28 | 13·12 | 14·02 | 14·97 | 15·98 | 17·05 | 18·18 | 19·37 | 20·63 | 21·97 |
| 20 | 23·38 | 24·86 | 26·43 | 28·09 | 29·83 | 31·67 | 33·61 | 35·65 | 37·80 | 40·05 |
| 30 | 42·43 | 44·92 | 47·55 | 50·31 | 53·19 | 56·23 | 59·41 | 62·75 | 66·25 | 69·92 |

| | |
|---|---|
| 40 | 73·76 |
| 50 | 123·3 |
| 60 | 199·2 |
| 70 | 311·6 |
| 80 | 473·4 |
| 90 | 701·0 |
| 100 | 1013·2 |
| 200 | 15544 |
| 300 | 85903 |

Critical temperature : 374°. Critical pressure : 220·6 bars. Interpolate logarithmically.

## *Composition of Fertilizers (Commercial Grades on the English Market)*

| | %N | | $\%P_2O_5$ | | | $\%K_2O$ |
|---|---|---|---|---|---|---|
| phate of ammonia . | . 20·6 | Superphosphate | . 16-18 | water soluble | Sulphate of potash . | 48 |
| trate of soda . . | . 16 | Triple superphosphate | 48 | | Muriate of potash . | 50-60 |
| trochalk . . . | . 15·5 | Basic slag . | . 10-18 | total | Potash manure . . | 40 |
| | | Rock phosphate | . 26-28 | | | |

ome of these fertilizers are mixtures of two or more constituents in naturally occurring or conventional proportions. 1952 values are given.

## *English Weights and Measures*

### *Length*

1 foot = 12 ins.   1 yard = 3 ft.
1 mile = 1760 yds. = 5280 ft.
(60 m.p.h. = 88 ft. per sec.)

1 surveyor's chain = 22 yds.
(1 engineer's chain = 100 ft.)
1 link = $\frac{1}{100}$ chain = 7·92 ins.
(1 rod, pole, or perch = $\frac{1}{4}$ chain.)
1 furlong = 10 chains = $\frac{1}{8}$ mile.

1 fathom = 6 ft.   1 cable = 100 fath.
1 nautical mile = 6080 ft.
= 1 minute of arc of equator.
(1 knot = 1 nautical mile per hour.)

Archaic (approximate) :
1 hand = 4 ins.   1 cubit = 18 ins.
1 ell = 45 ins.   1 league = 3 miles.

### *Area*

1 acre = 10 sq. chains=4840 sq. yds.
1 sq. mile = 640 acres.
(1 rod = $\frac{1}{160}$ ac., 1 rood = $\frac{1}{4}$ ac.)

### *Weight*

#### *Avoirdupois* (*General System*)

1 ounce (oz.) = 16 drams (dm.).
1 pound (lb.) = 16 oz.
1 stone = 14 lb.   1 quarter = 28 lb.
1 hundredweight (cwt.) = 112 lb.
= 4 quarters = 8 stones.
1 ton = 20 cwt. = 2240 lb.

#### *Apothecary* (*Drugs*)

1 drachm=60 grs.
(1 scruple=20 grs.)

#### *Troy* (*Precious Metals*)

1 pennyweight (dwt.) = 24 grs.

1 ounce = 20 dwt. = 8 drachms.
1 pound = 12 oz. = 5760 grs.
(1 lb. avoirdupois = 7000 grs.)

#### *Money*

1 shilling (s.) = 12 pence (d.).
1 pound (£) = 20s.
Also farthing (¼d.), florin (2s.), half-crown (2s. 6d.) and guinea (21s.).

### *Volume*

#### (*Liquid*)

1 gallon = 4 quarts = 8 pints
= 32 gills = ·16054 (1/6·2290) cu. ft
1 gallon water (62° F.) weighs 10 lb. av
(in air against brass weights).

1 fluid drachm (ʒ) = 60 minims (♏).
1 fluid oz. (℥) = 8 dr. = $\frac{1}{20}$ pint.
1 fluid oz. water (62° F.) weighs 1 oz. av

#### (*Dry*)

1 peck = 2 gals.   1 bushel = 8 gals.
1 sack = 4 bush.   1 quarter = 8 bush
(1 firkin = 9 gals.   1 barrel = 36 gals

1 register ton = 100 cu. ft.
1 rod = 1000 cu. ft.

## *Weights and Measures of the United States of America*

Weights and measures are the same as the English systems, except for small differences in standards, and the follc ing exceptions :—

1 hundredweight (short) = 100 lb., and 1 ton (short) = 20 cwt. = 2000 lb. The English (long) ton, 2240 lb.. also used.

Dry measures of volume are 3 per cent. less than the corresponding English measures, being based on the Winches bushel instead of the Imperial bushel. Fluid measures of volume are $\frac{5}{6}$ of the corresponding English measures. (Her in the U.S. system 1 bushel = 9·3092 fluid gallons.) Also 1 U.S. fluid ounce = $\frac{1}{16}$ U.S. pint, so that a fluid ounce water weighs 1·0408 oz. av. (See also conversion factors.)

A township = 36 sq. miles. A billion (French *Milliard*) = $10^9$, a trillion = $10^{12}$. (English billion = $10^{12}$, trilli = $10^{24}$.)

## *The Metric System*

The metre is (very nearly) $10^{-7}$ of the distance from the pole to the equator.

The gram is (very nearly) the mass of 1 cubic cm. of water at maximum density (3·98° C.). (Actual mass of 1 c of water = ·999973 g.)

The litre is (very nearly) 1000 c.c. (1 litre = the volume of 1000 g. of water at 3·98° C. = 1000·027 c.c.)

Also 1 are = 100 sq. metres, so that 1 hectare = 10,000 sq. m. = $\frac{1}{100}$ sq. km. ; 1 metric quintal (doppel zentn U.S.S.R. centner) = 100 kg. ; 1 metric ton = 1000 kg. ; 1 Continental pound = $\frac{1}{2}$ kg. ; 1 stere = 1 cu. m. 1 micr ($\mu$) = $10^{-6}$ m. 1 angstrom unit (Å) = $10^{-10}$ m.

Prefixes: micro-, $10^{-6}$; mille-, $10^{-3}$; centi-, $10^{-2}$; deci-, $10^{-1}$; Deka-, 10; Hecto-, $10^{2}$; Kilo-, $10^{3}$; Myria-, 1 Mega-, $10^{6}$.

## *Conversion Factors*

### *Basic Factors*

1 yard (G.B.) = ·914,399 metres.
1 yard (U.S.A.) = ·914,402 metres.
1 pound = 453·5924 grams.
1 gallon (G.B.) = 4·545,963 litres.
1 gallon (U.S.A.) = 3·785,332 litres.
1 bushel (U.S.A.) = 35·238,33 litres.

### *Capacity* (*Great Britain*)

1 pint = ·56825 (1/1·7598) litres.
1 gallon = 4·5460 (1/·21998) litres.
1 bushel = 36·368 (1/·027497) litres.

### *Length*

1 inch = 2·5400 (1/·39370) cm.
1 foot = ·30480 (1/3·2808) m.
1 yard = ·91440 (1/1·0936) m.
1 mile = 1·6093 (1/·62137) km.

### *Capacity* (*U.S.A.*)

Fluid
1 pint = ·47317 (1/2·1134) lit.
1 gall. = 3·7853 (1/·26418) lit.
Dry : 1 bush. = 35·238 (1/·028378) lit.

1 British pint, gallon, etc. = 1·20094 (1/·83268) U.S. fluid pints, gallons, etc.
1 British bushel, etc. = 1·03205 (1/·96895) U.S. bushels, etc. (dry).

### *Weight*

1 grain = ·064799 (1/15·432) g.
1 ounce = 28·350 (1/·035274) g.
1 pound = ·45359 (1/2·2046) kg.
1 cwt. = ·50802 (1/1·9684) m. quint
1 short cwt. = ·45359 (1/2·2046) m.
1 ton = 1·0160 (1/·98421) metric to
1 short ton = ·90718 (1/1·1023) m.t

### *Volume*

1 cu. inch = 16·387 (1/·061024) c.c.
1 cu. foot = ·028317 (1/35·315) cu.
1 cu. yard = ·76455 (1/1·3080) cu.m

### *Area, etc.*

1 sq. inch = 6·4516 (1/·15500) sq. cm.
1 sq. foot = ·092903 (1/10·764) sq. m.
1 sq. yard = ·83613 (1/1·1960) sq. m.
1 sq. mile = 2·5900 (1/·38610) sq. km.

1 acre = ·40468 (1/2·4711) hectares.
1 lb. per acre = 1·1209 (1/·89218) kg. per ha.
1 cwt. per acre = 1·2554 (1/·79659) m.q. per ha.

1 ton per acre = 2·5107 (1/·39829 tons per ha.
1 bushel per acre = 89·867 (1/·011 litres per ha. (U.S.A.) = 87· (1/·011484) litres per ha.

Bushel weights (average): wheat, 63 lb.; oats, 42 lb.; barley, 56 lb.; maize, 60 lb.; beans, peas, 64 lb.

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